Minimizing Aroma Loss

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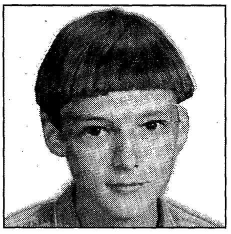
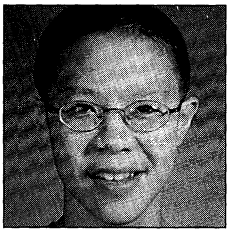


Robert Barrington Leigh is a 13-year-old student at Vernon Barford Junior High School in Edmonton, Canada. He has always been interested in mathematics and has competed in several contests. In grade 6 he won first place in the CNML, and in grade 7, he gained the Edmonton Junior High math trophy. Robert enjoys Professor A. Liu's math club, and it is under Professor Liu's guidance that he worked on this paper.



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Imagine that you are the owner of a small coffee shop, and you have just imported a box of the finest Colombian coffee beans. As you open it, you savor the aroma. Suddenly, your smile turns into a frown as you realize that some of the essence of the coffee has evaporated into thin air.

We use the following **mathematical model** to measure the loss. We assume that there are n kilograms of coffee beans initially, where n is a positive integer, and that you will use 1 kilogram each day. Each kilogram in a box loses 1 aroma point every time the box is opened. Fortunately, you have some empty boxes which help in reducing future losses. Let k be the number of boxes available, including the one in which the coffee beans come. You want to minimize the total number of points lost.

Let us first work out an example with $k = 2$ and $n = 6$. After checking all cases, we find this optimal strategy. Let the boxes be numbered 1 and 2.

Day	Open box	Points lost	Shift to Box 2	Amount in Box 1	Amount in Box 2
1	1	6	2 kg	3 kg	2 kg
2	2	2		3 kg	1 kg
3.	2	1		3 kg	
4	1	3	1 kg	1 kg	1 kg
5	2	1		1 kg	
6	1	1			
	Total =	14			

We now consider the general problem. Clearly, counting the number of points lost each day is not a promising approach, especially since we do not even know how many kilograms of coffee beans are to be transferred from which box to which, and when. The main idea behind our attack of this problem is to count the number of points lost by each kilogram.

The number of points each kilogram of coffee beans loses is equal to the number of times it is exposed. We keep track of this by putting a label on each kilogram. Number the boxes 1 to k. A label is initially empty. Every time the kilogram is exposed while in box i, add an i to the end of its current label. The label changes progressively until the kilogram is used up. Its length at that time is the total number of points lost.

Each label starts with a 1. By symmetry, we can arrange to have no more coffee beans in a box with a higher number than in a box with a lower one. Each day, we always open the non-empty box with the highest number. Thus we never transfer coffee beans from a box with a higher number to a box with a lower one. This means that the terms in each label are non-descending. Since exactly one kilogram of coffee beans is used each day, no two kilograms can have the same label. What we want is a set of the shortest n labels.

Let us return to our example with $k=2$ and $n=6$. There is only one label of length 1, namely 1. There are two labels of length 2 and three labels of length 3. They are 11, 12, 111, 112 and 122. Thus the minimum number of points lost is $1+2+2+3+3+3=14$. This justifies that our strategy is indeed optimal. In fact, it is the only one that leads to the optimal result, since the labels tell us precisely what to do.

Each kilogram is exposed in box 1 on day 1. The kilogram labeled 1 is used immediately. The kilograms labeled 12 and 122 must be shifted to box 2 then. They are used on days 2 and 3. The remaining three kilograms are all exposed in box 1 on day 4. The kilogram labelled 11 is used immediately, while the kilogram labeled 112 must be shifted to box 2. It is used on day 5, while the kilogram labeled 111 stays in box 1 throughout, and is used on day 6.

The general problem is solved if we can count the number of distinct labels of length l with non-descending terms such that the first is 1 and none exceeds k. As another example, consider the case $k=3$ and $l=5$. There are 15 such labels:

11111	11122	11222	11333	12233
11112	11123	11223	12222	12333
11113	11133	11233	12223	13333

Counting the labels directly is no easy matter either. We now change each into a binary sequence as follows. Write down a number of 0's equal to the number of 1's in the label. Insert a 1 after this block. Then write down a number of 0's equal to the number of 2's, followed by another 1, and so on. Note that each binary sequence consists of k 1's and l 0's, starts with a 0 and ends with a 1.

As an example, consider the label 11122. We start off with three 0's followed by a 1. Then we write down two 0's followed by a 1. Finally, since the label contains no 3's, we just write down one more 1, yielding the binary sequence 00010011. Conversely, consider the binary sequence 01000101. We start off with one 1, followed by three 2's and then one 3, yielding the label 12223. It is clear that each label is matched with a unique binary sequence whose first term is 0 and last term 1, and vice versa. The corresponding binary sequences are listed after the labels in the chart below.

11111	00000111	11133	00011001	12222	01000011
11112	00001011	11222	00100011	12223	01000101
11113	00001101	11223	00100101	12233	01001001
11122	00010011	11233	00101001	12333	01010001
11123	00010101	11333	00110001	13333	01100001

It is not too difficult to count such binary sequences. As noted before, they are of length $l+k$. Since the first term is always 0 and the last term is always 1, we only need to consider the $k+l-2$ terms in between. They consist of $l-1$ 0's and $k-1$ 1's, and all we have to do is count the number of ways of placing the 1's. The answer is the binomial coefficient $\binom{k+l-2}{k-1}$. When $k=3$ and $l=5$, $\binom{k+l-2}{k-1}=\binom{6}{2}=15$. Hence there are indeed 15 labels of length five, as we saw earlier.

For n kilograms of coffee beans, let the longest labels have length m. This means that we use all labels of length less than m, and as many labels of length m as needed to bring the total up to n. Hence m is the largest integer such that the total number N of labels of length from 1 to $m-1$ is less than n. Clearly, $N=\binom{k-1}{k-1}+\binom{k}{k-1}+\binom{k+1}{k-1}+\cdots+\binom{k+m-3}{k-1}=\binom{k+m-2}{k}$.

For any positive integer n, let m be the largest positive integer such that $n>\binom{k+m-2}{k}$. Let $r=n-\binom{k+m-2}{k}$, where $1\le r\le\binom{k+m-1}{k}-\binom{k+m-2}{k}=\binom{k+m-2}{k-1}$. Then the n labels consists of $\binom{k-1}{k-1}$ of length 1, $\binom{k}{k-1}$ of length $2,\ldots,\binom{k+m-3}{k-1}$ of length $m-1$, and r of length m. It follows that the minimum number of points lost is

$$\binom{k-1}{k-1}1+\binom{k}{k-1}2+\cdots+\binom{k+m-3}{k-1}(m-1)+rm,$$

and that this optimal value can be attained.

In our original example, $n=6$ and $k=2$. Now m is determined by $6>\binom{m}{2}$, so that $m=3$. Hence $r=6-\binom{3}{2}=3$ and the minimum number of points lost is $\binom{1}{1}1+\binom{2}{1}2+\binom{3}{1}3=14$, as we found. If $k=3$, then $m=3$, $r=2$ and 13 points are lost. If $k=4$, then $m=3$, $r=1$ and 12 points are lost. If $k=5$, then $m=2$, $r=5$ and 11 points are lost. This is the best that can be done with $n=6$, and we leave to the reader the details of how to move the kilograms of coffee around.

Big Bucks

The sharp eye of Robert Beezer (University of Puget Sound, beezer@ups.edu) noticed the following item in the May 4, 1999 *Christian Science Monitor*:

> **THEIR PROFIT: A MERE 14,285%** Internet stocks are fetching hefty prices these days. Then there's the investment that former Albuquerque, N.M. broker Eric Wade and a friend **paid $70** for in 1994...and sold late last month for a cool $1 million.

As Professor Beezer says, "What's a couple of decimal places among friends?"