Lorentzian Geometry Outside of General Relativity: an Application to Airline Boarding

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1 Introduction

Two-dimensional Lorentzian geometry has recently found application in some models of non-relativistic systems, most profitably for the process of boarding an aeroplane. The duration of the boarding process is assumed to be a result of passengers standing in the isle and blocking each others' way. According to this model, the expected boarding time is equal to the length of the longest timelike geodesic on a certain Lorentzian manifold. Although the asymptotic approach used is valid only for large numbers of passengers, it is useful in comparing the effectiveness of various airline policies such as asking passengers with seats in the back half of the plane to board first.

To explain the ideas involved, we first treat a related but simpler question, stemming from the idea of the longest increasing subsequence of a permutation. A description of the boarding model is followed by its translation into Lorentzian geometry and a sample calculation. Finally, we discuss the validity and practical implications of the model. Many of the theorems we will quote have probabilistic statements, which we will not attempt to prove because the methods involved are not relevant to general relativity. Instead we will often provide a heuristic argument.

2 Longest Increasing Subsequences in Permutations

Suppose that σ is a permutation of $1, \dots, n$. Let $L(\sigma)$ be the longest increasing subsequence of the sequence $\sigma(1), \dots, \sigma(n)$. It is possible that $L(\sigma) = n$—when σ is the identity—or that $L(\sigma) = 1$—when the sequence is decreasing. We wish to describe approximately the expected value of $L(\sigma)$, with σ chosen randomly from S_n, when n is large.

This problem can be reworded in terms of a Lorentzian geometry on the unit square $A = [0, 1]^2$. Pick n points in A at random, and label them $P_1, \dots, P_n$ in increasing order of x coordinate. For each i, let $\sigma(i)$ be the position of P_i when the points are arranged in order of y coordinate. This defines a permutation $\sigma \in S_n$ (unless two points lie on the same horizontal or vertical line, which happens with probability 0), which by symmetry must be uniformly distributed in S_n. We impart A with the metric

$$ds^2 = -4n dx\, dy,$$

which is the Minkowski metric in rotated, scaled coordinates. The null geodesics in this geometry are horizontal and vertical lines, and for each point the upper right light cone is taken as the causal future. In its new guise, an increasing subsequence is a subset of points that are pairwise causally related: for every two points in the set, one of them has the greater x coordinate as well as the greater y coordinate. Call such a subset a *chain*. We ask how many points are in a largest chain, on average. The answer is based on a result discovered by Vershik and Kirov, and also independently by Logan and Shepp (see references in [1]). We adopt the notation *with high probability* to assert that the probability of an event tends to 1 as $n \to \infty$.

Proposition 1. *Let n points be chosen with uniform probability in $[0,1]^2$, and let K be the size of the largest chain. Then for each $\varepsilon > 0$, with high probability,*

$$\left|K - 2\sqrt{n}\right| < \varepsilon\sqrt{n}.$$

Moreover, any chain of points (x_i, y_i) of size K satisfies

$$|y_i - x_i| < \varepsilon$$

for all i with high probability.

Our as yet tenuous claim is that asymptotically, the behaviour of large chains can be understood by looking at geodesics. Specifically, there is a unique longest geodesic in our space, namely the straight line joining $(0,0)$ to $(1,1)$, and its length (total proper time) is $2\sqrt{n}$. The proposition asserts that points in a largest chain tend to lie close to this geodesic, and that their number is equal to its length. This analogy can be generalized to other metrics, as we will see.

3 The Airline Boarding Model

The main model of interest describes the process of boarding an aeroplane in a busy airport [2]. Passengers often spend most of this time waiting for each other, and it would be beneficial to optimize the process. For example, at boarding time, suppose that the n passengers queue in order of assigned seat: the first person in line will occupy a front seat, and so on. The first passenger may begin to stow his luggage immediately, but subsequent passengers must wait for everybody in front of them before even reaching their seats. We assume that each passenger incurs the same delay time D between reaching her assigned seat and sitting down; during this time she is blocking passengers behind her. The *boarding time* is the time for all passengers to sit down, thus it is a multiple of D.

Now suppose that the passengers are queued in the reverse order, with the first person in line sitting at the back of the aeroplane. One might expect that everybody may now walk onto the aeroplane and take their seats concurrently; however, in reality they will not all fit in the aisle at once. Let u be the average length of aisle space occupied by one person, and let w be the average distance between seats (i.e. the length of the aisle divided by n in a full aeroplane). Then this arrangement of passengers will be efficient only if $u < w$. Define the nondimensional *crowding parameter* $k = u/w$, whose crucial impact on the model behaviour we can already begin to see.

The purpose of this model is to assess boarding policies—schedules of announcements of the form "Passengers in rows 30-40 may now join the line for boarding"—and whether they encourage an efficient boarding process. Given such a policy, and knowing how passengers respond to it, we can concoct a probability density $P(q, r)$ relating the location q of a passenger in the queue to her row r in the aeroplane. We work in the limit of large n, and normalize both q and r to $[0, 1]$. Now $P(q, r)$ must be a nonnegative, continuous function on $[0, 1]^2$ such that

$$\int_0^1 P(q,r)dr = 1 \quad \text{for all } q\text{, i.e. there is one person at each position in the queue;}$$

$$\int_0^1 P(q,r)dq = 1 \quad \text{for all } r\text{, i.e. there is one person in each seat.}$$

For our model with large n, we will take the passengers to be n points in $[0, 1]^2$, chosen randomly according to $P(q, r)$, Note that $P(q, r)$ is normalized independently of the number of passengers, in contrast with the metric of the previous section which contained n explicitly.

In short, we have the following parameters:

$P(q, r)$	probability density relating passengers' queue positions and assigned rows
n	number of passengers, assumed large
D	delay incurred by a passenger stowing luggage and preparing to sit down
k	aisle space occupied by one person during the delay, in units of seats (assuming that seats are equally spaced in one column)

Before introducing a metric on $[0, 1]^2$ we first define a partial order called *blocking*. Suppose that after stowing his luggage, a certain passenger A lingers in the aisle next to his seat, but never sits down. Then those who are in the queue behind A and are assigned seats beyond A will never reach their places; we shall say that A *directly blocks* them. Even some of those who wish to sit in front of A may be detained by others who are waiting. In our notation, A *blocks* all these people who are waiting. The blocking relation forms a partial ordering on all passengers, i.e. it is transitive. Again, we define a *chain* as a sequence of passengers each of whom blocks the next.

Proposition 2. *Let K be the size of the longest chain of passengers. Then the boarding time is DK.*

Proof. Let DT be the boarding time (it must be a multiple of D). We have $T \geq K$ because among the members of the chain of length K, only one may sit down at a time. On the other hand, we can always construct a chain of length K, as follows. Let A_1 be a passenger who sat down last. Inductively, for $2 \leq i \leq T$, let A_i be a passenger who was blocking A_{i-1} just before A_{i-1} became free to sit down. Thus A_i sat down no earlier than time $D(T-i)$. This process constructs a chain $\{A_1, \ldots, A_T\}$ of length T. □

4 Lorentzian description

In the large-n limit, we can determine whether A is likely to block B using only the two pairs of coordinates, respectively (q_A, r_A) and (q_B, r_B). Let $\Delta q = q_B - q_A$ and $\Delta r = r_B - r_A$. Assume that these coordinates are very close on the unit square. As a first, necessary condition for A to block B, we require that A stands before B in the queue: $q_A < q_B$. Let $C_1, \ldots, C_I$ be the intermediate passengers between A and B that A blocks directly, that is passengers such that $q_A < q_{C_1} < \cdots < q_{C_I} < q_B$ and $r_{C_i} > r_A$. The expected number of such passengers, I, is

$$\begin{aligned} E[I] &= n \int_{q_A}^{q_B} \int_{r_A}^{1} P(q,r)\, dr\, dq \\ &\approx n(\Delta q) \int_{r_A}^{1} P(q_A, r)\, dr \end{aligned}$$

Define

$$\alpha(q_0, r_0) = \int_{r_0}^{1} P(q_0, r)\, dr$$

so that $E[I] = n\alpha(q_A, r_A)\Delta q$. If we wanted to include all passengers between A and B who are blocked by A, directly or indirectly, then we would have to change the lower bound r_A to a smaller number depending on $q - q_A$. However, the contribution of this change is small, since there are only $n\Delta q$ people queued between A and B. Thus, we take the expression found for $E[I]$ to be a good estimate for the number of blocked passengers between A and B. Since n is large, we will assume that I achieves exactly this value. Accordingly, B will be blocked if and only if $\Delta q > 0$ and

$$\begin{aligned} \Delta r &\geq -\frac{kI}{n} \\ &= k\alpha(q_A, r_A)\Delta q \end{aligned}$$

This is valid only locally, for close A and B, but in fact the blocking relation will only be important locally, so we will always use this formula.

We are now ready to introduce a metric on $[0,1]^2$ (with the obvious coordinates (q, r)):

$$ds^2 = -4P(q,r)(dq dr + k\alpha(q,r) dq^2)$$

with α defined above in terms of P. The null directions for this metric are $(0,1)$ and $(1, -k\alpha(q,r))$, with space-like vectors confined to a subset of the second and fourth quadrants of tangent space. Let us assign the light cone including the first quadrant to be the causal future, and the other light cone to be the causal past. Revisiting the asymptotic description we established for the blocking relation, we find that w.h.p. A blocks B if and only if B lies in the causal future of A. Therefore our goal of finding the longest chain in the blocking partial order is equivalent to finding the largest set of causally related points, or *chain*, a task that we began to examine in the first section. There is one other aspect of the metric that we have not yet justified: the scaling factor $4P(q,r)$. The volume 2-form associated with the metric is

$$\sqrt{|\det g|}dq \wedge dr = 4P(q,r)dq \wedge dr,$$

which is proportional to the probability density.

Our task translates to the following: Given the manifold $M = [0,1]^2$ with a metric g, find the expected length of the longest chain when n points are chosen in M according to a probability distribution proportional to the volume form associated to g.

5 Conformal Flatness

A Lorentzian manifold M with metric g_{ab} is called *conformally flat* if for each point $p \in M$ there is a coordinate system x^μ defined in a neighbourhood U of p, and a scalar function $\sigma\colon U \to (0,\infty)$, such that

$$g_{\mu\nu} = \sigma\eta_{\mu\nu}.$$

We can interpret this in the following way: if we multiply the metric on M by a scalar function σ, then M becomes locally isometric to the Minkowski plane. It is a remarkable fact that every two-dimensional (pseudo)-Riemannian manifold is conformally flat; this is not true in higher dimensions.

Proposition 3. *Every two-dimensional Lorentzian manifold M is conformally flat.*

Proof. [5] Let $p \in M$, and choose an arbitrary chart $\phi\colon U \to V \subset \mathbb{R}^2$, where U is a neighbourhood of p. Let g_{xx}, g_{xy}, g_{yy}, $|g|$ denote the components and determinant of the pushed-forward metric on V, respectively; equivalently, express the metric in the coordinates associated with ϕ. Consider the following differential 1-forms, ω_+ and ω_-, defined on V:

$$\omega_\pm(x,y) = \sqrt{g_{xx}(x,y)}dx + \frac{g_{xy}(x,y) \pm \sqrt{|g(x,y)|}}{\sqrt{g_{xx}(x,y)}}dy$$

We claim that there exist functions $\lambda_\pm$ such that $\omega_\pm\lambda_\pm$ is a full derivative, say $\omega_\pm\lambda_\pm = du_\pm$. For generic choices of coordinates, the components of $\omega_\pm$ will not vanish in a neighbourhood of p. Then the theory of differential equations provides us with integrating factors $\lambda_\pm$. Generically, $u_\pm$ will be independent and hence can serve as new local coordinates. Now

$$\begin{aligned}
\frac{du_+\,du_-}{\lambda_+\lambda_-} &= \left(\sqrt{g_{xx}(x,y)}dx + \frac{g_{xy}(x,y)+\sqrt{|g|\,(x,y)}}{\sqrt{g_{xx}(x,y)}}dy\right)\left(\sqrt{g_{xx}(x,y)}dx + \frac{g_{xy}(x,y)-\sqrt{|g|\,(x,y)}}{\sqrt{g_{xx}(x,y)}}dy\right) \\
&= \left(\sqrt{g_{xx}}dx + \frac{g_{xy}}{\sqrt{g_{xx}}}dy\right)^2 - \frac{|g|}{g_{xx}}dy^2 \\
&= g_{xx}dx^2 + 2g_{xy}dx\,dy + \frac{g_{xy}^2}{g_{xx}}dy^2 - \frac{g_{xy}^2 - g_{xx}g_{yy}}{g_{xx}}dy^2 \\
&= g_{xx}dx^2 + 2g_{xy}dx\,dy + g_{yy}dy^2 \\
&= ds^2
\end{aligned}$$

Taking u_+ and u_- as coordinates on M, we have established that the metric is $(\lambda_+\lambda_-)^{-1}du_+\,du_-$ which is recognizable as the metric of Minkowski space in coordinates rotated by 45°. If desired, we may recover the Minkowski metric (with a scaling factor proportional to $(\lambda_+\lambda_-)^{-1}$) by changing coordinates once more to $(u_+ + u_-, u_+ - u_-)$. □

The value of this theorem is that it reduces our question about chains in a bizarre partial order to one about the well-studied topic of longest increasing subsequences, the type of chain we examined in the first section. Although it is not part of the theorem, we will assume that there is a single chart (u_+, u_-) for the whole manifold, on which the metric is a multiple of $du_+\,du_-$. For any given metric, we could in principle verify this. Since the probability density in our problem is proportional to the volume form of the metric, the total probability in a given volume is preserved under isometry. Therefore the probability density to use when studying the manifold in (u_+, u_-) coordinates is proportional to the scaling factor $(\lambda_+\lambda_-)^{-1}$ in the new metric.

6 Relationship to Geodesics

The relationship between chains and geodesics generalizes to an arbitrary metric that is a multiple of the Minkowski metric. We quote the following theorem originally from [4], adapted to the language of Lorentzian geometry.

Theorem 1. *Let $P(x,y)$ be a continuous density distribution on the unit square and let S be a set of n points in the unit square chosen with respect to Q. Denote by K the largest increasing subset of S. Consider the metric $-4P(x,y)dx\,dy$ on the unit square, and let the nondecreasing function $y = \phi(x)$ have longest proper time among all such curves in the unit square with boundary conditions $\phi(0) = 0$, $\phi(1) = 1$.*

1. *For all $\varepsilon > 0$, with high probability $|K - L\sqrt{n}| < \varepsilon\sqrt{n}$, where L is the proper time of $y = \phi(x)$.*
2. *For any $\varepsilon, \delta > 0$, with high probability an increasing subset of size $K - \varepsilon\sqrt{N}$ can be found in a δ neighbourhood of $y = \phi(x)$.*

Proof. Consider any nondecreasing function $y = \psi(x)$, $\psi\colon [0,1] \to [0,1]$, satisfying $\psi(0) = 0$ and $\psi(1) = 1$. Let $\delta = 1/N$ for some large N, and consider the following subset of the unit square:

$$B = \{(x,y) \mid (i-1)\delta \le x \le i\delta, \phi((i-1)\delta) \le y \le \phi(i\delta), \text{ for some } i = 1,2,\ldots,N\}$$

Note that B is a union of causally related rectangles (blocks). It is also somewhat like a δ neighbourhood of $y = \phi(x)$. Approximating $P(x,y)$ to be constant on each block, we expect the block containing $(x,\phi(x))$ to contain $E = n\delta^2\phi'(x)P(x,\phi(x))$ uniformly distributed points of S. By the result in the first section, the longest increasing sequence in that block has length $2\sqrt{E}$ with high probability, but this is exactly the Lorentzian length of the diagonal line connecting $(x,\phi(x))$ to $(x+\delta,\phi(x+\delta)$. Thus the total length of an increasing sequence in $S \cap B$ is, with high probability, the proper time of $y = \phi(x)$. With this fact, the theorem at least becomes plausible. □

It is not essential in the theorem to use a square domain, because we can choose $P(x,y)$ to be very small outside the domain of interest. Similarly, we can choose different starting and ending points instead of $(0,0)$ and $(1,1)$, by adjusting $P(x,y)$ appropriately. Thanks to this theorem, the length of a longest chain is a purely geometrical property of the manifold, and we can return to (q,r) coordinates. In short, the boarding time is, with high probability, equal to the length of the longest timelike curve (often a geodesic).

7 Example

The simplest boarding policy to analyze is the trivial one whereby passengers are asked to queue randomly. This assures a uniform distribution $P(q,r) = 1$, yielding the associated metric

$$ds^2 = -4(dq\,dr + k(1-r)dq^2)$$

on $[0,1]^2$. An easy computation of the Christoffel symbols,

$$\begin{aligned} {\Gamma^q}_{qq} &= k \\ {\Gamma^r}_{qq} &= 2k^2(r-1) \\ {\Gamma^r}_{qr} = {\Gamma^r}_{rq} &= -k, \end{aligned}$$

enables us to write the equations defining a geodesic $(q(\lambda), r(\lambda))$:

$$\begin{aligned} \frac{d^2q}{d\lambda^2} + k\left(\frac{dq}{d\lambda}\right)^2 &= 0, \\ \frac{d^2r}{d\lambda^2} + 2k^2(r-1)\left(\frac{dq}{d\lambda}\right)^2 - 2k\frac{dq}{d\lambda}\frac{dr}{d\lambda} &= 0. \end{aligned}$$

The first equation is independent of $r(\lambda)$, so its general solution can be written down immediately.

$$\frac{dq}{d\lambda} = \frac{1}{k(\lambda_0 - \lambda)}$$
$$\Rightarrow q(\lambda) = -\frac{1}{k}\log|\lambda_0 - \lambda| + q_0,$$

with arbitrary real constants λ_0 and q_0. The differential equation for $r(\lambda)$ can be treated by reparameterizing r as a function of q: we are seeking timelike, causally-oriented geodesics, for which q must be an increasing function of λ. Since $\frac{d}{d\lambda} = \frac{dq}{d\lambda}\frac{d}{dq}$, the differential equation simplifies to

$$\left(\frac{d^2r}{dq^2} - k\frac{dr}{dq}\right) + 2k^2(r-1) - 2k\frac{dr}{dq} = 0$$
$$\Leftrightarrow \frac{d^2r}{dq^2} - 3k\frac{dr}{dq} + 2k^2 r = 2k^2.$$

A particular solution is $r \equiv 1$, while the homogeneous solution can be read off from the characteristic polynomial $t^2 - 3kt + 2k^2 = (t-k)(t-2k)$.Thus

$$r(q) = 1 + C_1 e^{kq} + C_2 e^{2kq}.$$

The geodesic of interest starts at $(0,0)$ and ends at $(1,1)$, thus there are boundary conditions

$$1 + C_1 + C_2 = 0,\ C_1 e^k + C_2 e^{2k} = 0$$
$$\Rightarrow C_1 = \frac{-e^k}{e^k - 1},\ C_2 = \frac{1}{e^k - 1}$$

However, for some values of k the function $r(q)$ veers outside of $[0,1]$. The equation for $r(q)$ is quadratic in e^{kq}, so that it is easy to study this behaviour: we find that $r(q)$ is viable provided $k \leq \log 2$. In this case, the desired length of the geodesic is

$$\begin{aligned} L(k) &= \int_{q=0}^{q=1} \sqrt{4\left(\frac{dq}{d\lambda}\frac{dr}{d\lambda} + k(1-r)\left(\frac{dq}{d\lambda}\right)^2\right)}\, d\lambda \\ &= \int_0^1 2\sqrt{\frac{dr}{dq} + k(1-r)dq} \\ &= 2\int_0^1 \sqrt{C_1 k e^{kq} + 2kC_2 e^{2kq} + k\left(-C_1 e^{kq} - C_2 e^{2kq}\right)} dq \\ &= 2\sqrt{\frac{C_2}{k}}\left(e^k - 1\right) \\ &= 2\sqrt{\frac{(e^k - 1)}{k}} \end{aligned}$$

For larger values of k, the longest timelike curve from $(0,0)$ to $(1,1)$ must be some union of geodesics inside the open square $(0,1)^2$, along with some segments of its boundary. Furthermore, a maximizing curve must be continuously differentiable, so the interior geodesics must be tangent to the sides of the rectilinear segments where they meet. Consideration of $r(q)$ as a quadratic function leads to a unique possibility: there is some critical value q_c, $0 < q_c < 1$, such that

$$r(q) = \begin{cases} 0 & q \leq q_c \\ 1 + C_1 e^{kq} + C_2 e^{2kq} & q > q_c \end{cases}$$

We now have three boundary conditions: $r(q)$ must pass through $(1,1)$ and $(q_c, 0)$, and its derivative must be continuous at $(q_c, 0)$. We find

$$q_c = 1 - \frac{\log 2}{k},\ C_2 = 4e^{-2k},\ C_1 = -4e^{-k}$$

The length of this curve may now be computed as

$$\begin{aligned} L(k) &= \left(\int_0^{q_c} + \int_{q_c}^1\right) 2\sqrt{\frac{dr}{dq} + k(1-r)dq} \\ &= 2q_c\sqrt{k} + 2\sqrt{\frac{C_2}{k}}\left(e^k - e^{kq_c}\right) \\ &= 2\left(\sqrt{k} - \frac{\log 2}{\sqrt{k}}\right) + 4\sqrt{\frac{(e^{-k})}{k}}\left(\frac{1}{2}e^k\right) \\ &= 2\sqrt{k} + \frac{2(1-\log 2)}{\sqrt{k}} \end{aligned}$$

In brief, the expected time for n passengers to board when they queue in a random order is

$$\begin{cases} 2D\sqrt{n}\sqrt{\frac{(e^k-1)}{k}} & 0 \leq k \leq \log 2 \\ 2D\sqrt{n}\left(\sqrt{k} + \frac{(1-\log 2)}{\sqrt{k}}\right) & k > \log 2 \end{cases}$$

8 Results

The authors of [3] derived the expected boarding time under 25 different passenger distributions $P(q,r)$. They assumed a realistic value of the crowding parameter: $k = 4$. Previous studies had analyzed these boarding policies via detailed simulation of the boarding process, including many minutiae not considered here, such as the finite walking speed of passengers. That the Lorentzian model agrees well–up to a proportionality factor–with the results of the simulations suggests that we have isolated the key features of the process. Moreover, in some cases the authors were able to leverage results about increasing sequences to estimate the discrepancy between this model, which is asymptotically valid for large n, and the simulations, which involve only 100–200 passengers. We will not describe this work which involves random matrix theory.

The implications of the study are surprising. Boarding policies that encourage passengers at the back of the aeroplane to board first, which are commonly used in practise, were found to be ineffective. To be precise, these policies are preferable for small values of k, but if $k > 1$ then they actually promote blocking. Suppose that passengers follow the policy exactly, and $P(q, r)$ is supported on the line joining $(0, 1)$ to $(1, 0)$. If $k < 1$ then no two points on this line are causally related; if $k > 1$ then all n passengers form a chain of causally related events. For realistic k, only one or two passengers can sit down at a time, which is obviously inefficient. Instead of the status quo boarding policy, the authors favour the trivial policy because it is almost optimal "among row dependent policies which do not severely constrain passengers." There is one substantial caveat: a boarding policy that asks window-seat passengers to board first, middle-seat passengers second, and isle-seat passengers third, will easily outperform random boarding because nobody has to stand up again after taking a seat. Effectively, this can be modelled by decreasing the delay time D.

References

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