An Introduction to Gelfand Pairs of Finite and Compact Groups

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1 General Approach for Finite Groups

A representation ρ of a finite group G is called **multiplicity-free** if the decomposition of ρ into irreducibles has no repetitions. We will demonstrate that in certain cases it is possible to verify this property even if the irreducible constituents of ρ are unknown. There is a natural algebraic translation of this definition:

Lemma 1. *Let ρ be a finite-dimensional representation of a finite group G. Then ρ is multiplicity-free if and only if the intertwining algebra $\operatorname{Hom}_G(\rho,\rho)$ is commutative. In this case, $\dim \operatorname{Hom}_G(\rho,\rho)$ is equal to the number of irreducible constituents of ρ.*

Proof. This was a homework exercise. Since ρ is completely reducible, suppose it has the decomposition

$$\rho = \bigoplus_{i=0}^{m} \rho_i,$$

possibly with repetitions. Let us sort the sequence $\rho_1, \rho_2, \ldots, \rho_m$ so that any equivalent representations occur in consecutive terms. Now an intertwining operator $A \in \operatorname{Hom}_G(\rho,\rho)$ can be considered as an $m \times m$ matrix of homomorphisms, with entries $A_{ij} \in \operatorname{Hom}_G(\rho_j, \rho_i)$. Schur's lemma tells us that

$$\operatorname{Hom}_G(\rho_j,\rho_i) \simeq \begin{cases} \mathbb{C} & \text{if } \rho_j \simeq \rho_i \\ \{0\} & \text{if } \rho_j \not\simeq \rho_i \end{cases}$$

We see that $\operatorname{Hom}_G(\rho,\rho)$ can be identified with an algebra of block-diagonal matrices over $\mathbb{C}$, where the block sizes are the multiplicities of equivalent ρ_i's. Composition of intertwining operators A corresponds to matrix multiplication. Thus $\operatorname{Hom}_G(\rho,\rho)$ is commutative if and only if all the blocks have size 1, i.e. the ρ_i are distinct. When this happens, the matrix algebra is simply the set of diagonal $m \times m$ matrices, so that $\dim \operatorname{Hom}_G(\rho,\rho) = m$. □

We will be concerned with the case that ρ is induced from a finite-dimensional representation (π, V) of a subgroup $H < G$; if $\rho = i_H^G \pi$ is multiplicity-free then we call (G, H, π) a **Gelfand triple**. In particular, if $\pi = 1_H$ is the trivial representation then H is called a **Gelfand subgroup** of G, and we say that (G, H) is a **Gelfand pair**.

Recall that the **Hecke algebra** $\mathcal{H} = \mathcal{H}(G,\pi)$ was defined as a space of functions that mimic the action of π on the left and right:

$$\mathcal{H} = \{\varphi\colon G \to \operatorname{End}_{\mathbb{C}}(V) \mid \varphi(kgh) = \pi(k)\circ\varphi(g)\circ\pi(h) \;\; \forall g \in G, k, h \in H\}.$$

Multiplication in the Hecke algebra is by convolution: for $\varphi_1, \varphi_2 \in \mathcal{H}$,

$$(\varphi_1 * \varphi_2)(g) = |H|^{-1} \sum_{g_0 \in G} \varphi_1(g_0) \circ \varphi_2(g_0^{-1} g).$$

Lemma 2. *With notation as above, the $\mathbb{C}$-algebras $\operatorname{Hom}_G(i_H^G\pi, i_H^G\pi)$ and $\mathcal{H}$ are isomorphic*

Proof. In class we established a vector space isomorphism. That this isomorphism respects multiplication was left as an exercise. ☐

Lemma 3. *Let $H < G$ be finite groups, π be a representation of H, and $\mathcal{H} = \mathcal{H}(G, \pi)$. Then $i_H^G\pi$ is multiplicity-free if and only if $\mathcal{H}$ is commutative. In this case, the number of irreducible constituents of $i_H^G\pi$ is $\dim \mathcal{H}$.*

Proof. This is an amalgamation of the two preceding lemmas. ☐

We present a tool to show that $\mathcal{H}$ is commutative. A map $\iota\colon G \to G$ is called an **involution** if $\iota^2 = \text{id}$ and $\iota(g_1g_2) = \iota(g_2)\iota(g_1)$ for $g_1, g_2 \in G$. Such an ι is evidently not required to be a homomorphism. Similarly, a linear map $\tilde{\iota}\colon \mathcal{H} \to \mathcal{H}$ is called an **involution** if $\tilde{\iota}^2 = \text{id}$ and $\tilde{\iota}(\varphi_1\varphi_2) = \tilde{\iota}(\varphi_2)\tilde{\iota}(\varphi_1)$ for $\varphi_1, \varphi_2 \in \mathcal{H}$.

Lemma 4. *Let $H < G$ be finite groups, (π, V) be a one-dimensional representation of H, and $\mathcal{H} = \mathcal{H}(G, \pi)$. Suppose that $\iota\colon G \to G$ is an involution, fixing H, such that $\pi(\iota(h)) = \pi(h)$ for all $h \in H$. Then the map $\tilde{\iota}\colon \mathcal{H} \to \mathcal{H}$,*

$$\tilde{\iota}(\varphi)(g) = \varphi(\iota(g)),$$

is an involution of $\mathcal{H}$.

Proof. There are three things to show: that $\tilde{\iota}(\varphi) \in \mathcal{H}$ for all $\varphi \in \mathcal{H}$, that $\tilde{\iota}^2 = \text{id}$, and that $\tilde{\iota}$ reverses multiplication. All are straightforward, but the latter requires $\operatorname{End}_{\mathbb{C}}(V)$ to be commutative, which is true only if $\dim V = 1$. ☐

Theorem 1. *Suppose that there is an involution $\iota\colon G \to G$ such that $HgH = H\iota(g)H$ for all $g \in G$. Then (G, H) is a Gelfand pair.*

Proof. We wish to show that $i_H^G\pi$ is multiplicity-free, where $\pi = 1_H$. Construct $\tilde{\iota}$ according to Lemma 4; note that the requirement $\pi(\iota(h)) = \pi(h)$ is vacuous in this context.

Now $\mathcal{H}$ consists of all function $\varphi\colon G \to \mathbb{C}$ such that φ is constant on H-H double cosets; thus, $\tilde{\iota}$ acts by the identity on $\mathcal{H}$. Finally, observe that $\tilde{\iota} = id$ being an involution implies that $\mathcal{H}$ is commutative. ☐

The last three results may be considered a recipe for identifying Gelfand pairs and Gelfand triples, one that we will apply in Sections 4 and 3.

2 Generalization to Compact Groups

Let G be a compact group with closed subgroup H, and let μ be a normalized Haar measure on G. To generalize our multiplicity-free criterion, we will not follow an analogous line of reasoning; rather, we will prove directly that if the appropriate generalization of the Hecke algebra is commutative, then H is a Gelfand subgroup of G. First, we introduce the necessary definitions:

We avoid defining Gelfand subgroups in terms of induced representations, because these may be infinite-dimensional. However, Frobenius reciprocity implies an equivalent definition in terms of restricted representations, which we will use for compact groups: A closed subgroup H of G is a **Gelfand subgroup**, or (G, H) is a **Gelfand pair**, if for every irreducible representation (ρ, V) of G, the subspace of H-fixed vectors, V^H, is at most one-dimensional.

Lemma 5. *If $H < G$ are finite groups then $i_H^G 1$ is multiplicity-free if and only if for every irreducible representation (ρ, V) of G, the subspace V^H is at most one-dimensional.*

Proof. Observe that $V^H \simeq \operatorname{Hom}_H(1_H, V)$. By Frobenius reciprocity, the latter has dimension

$$\left\langle \chi_{1_H}, r_G^H \chi_\rho \right\rangle_H = \left\langle \chi_{i_H^G 1}, \chi_\rho \right\rangle_G,$$

which is the multiplicity of ρ in $i_H^G 1$. This quantity is at most 1 for every irreducible ρ, if and only if $i_H^G 1$ is multiplicity-free. ☐

Denote by $C(G)$ the set of continuous functions $\varphi\colon G \to \mathbb{C}$. This forms an algebra (without a multiplicative identity) under the convolution:

$$\varphi_1 * \varphi_2(g) = \int_{g_0 \in G} \varphi_1(g_0)\varphi_2(g_0^{-1}g)d\mu$$

Identify the subalgebra

$$\mathcal{H} = \mathcal{H}(G,H) = \{\varphi \in C(G) \mid \varphi(hg) = \varphi(g) = \varphi(gh)\ \forall g \in G, h \in H\}.$$

In the finite group setting, a representation (π, V) naturally gives rise to an action of the group algebra on V. Analogously, given a representation (π, V) of G and a function $\varphi \in C(G)$, define $\pi(\varphi)\colon V \to V$ by

$$\pi(\varphi)v = \int_{g \in G} \varphi(g)\pi(g)v d\mu.$$

Lemma 6. $\pi(\varphi_1 * \varphi_2) = \pi(\varphi_1) \circ \pi(\varphi_2)$

Proof.

$$\begin{aligned}
\pi(\varphi_1 * \varphi_2)v &= \int_{g \in G}\int_{g_0 \in G} \varphi_1(g_0)\varphi_2(g_0^{-1}g)\pi(g)v d\mu(g_0)d\mu(g) \\
&= \int_{g' \in G}\int_{g_0 \in G} \varphi_1(g_0)\varphi_2(g')\pi(g_0g')d\mu(g_0)d\mu(g') \\
&= \int_{g_0 \in G} \varphi_1(g_0)\pi(g_0)\int_{g' \in G} \varphi_2(g')\pi(g')v d\mu(g')d\mu(g_0) \\
&= \pi(\varphi_1)\left(\pi(\varphi_2)v\right)
\end{aligned}$$

□

The following theorem generalizes Lemma 3.

Theorem 2. *Let G be a compact group and $H < G$ be closed. If the convolution algebra $\mathcal{H}(G,H)$ is commutative, then H is a Gelfand subgroup of G.*

Proof. Let (π, V) be an irreducible representation of G; thus $n = \dim V < \infty$. Suppose that the subspace V^H is at least two-dimensional; we will construct two elements of $\mathcal{H}$ that do not commute. Recall that there is a G-invariant inner product $\langle\ ,\ \rangle$ on V. Thus there is an orthonormal basis $\{v_i\}_{i=1}^n$ of V such that $\operatorname{span}\{v_1, v_2\} \subset V^H$, i.e. $\pi(h)v_1 = v_1$ and $\pi(h)v_2 = v_2$ for $h \in H$. Define $\varphi \in C(G)$ in either of two equivalent ways:

$$\begin{aligned}
\varphi(g) &= \langle v_2, \pi(g)v_1\rangle \\
&= \left\langle \pi(g)^{-1}, v_1\right\rangle.
\end{aligned}$$

Observe that $\varphi \in \mathcal{H}$ because v_1 and v_2 are H-fixed: explicitly, for $h \in H$,

$$\varphi(gh) = \langle v_2, \pi(gh)v_1\rangle = \langle v_2, \pi(g)\pi(h)v_1\rangle = \langle v_2, \pi(g)v_1\rangle = \varphi(g),$$

and

$$\varphi(hg) = \left\langle \pi(hg)^{-1}v_2, v_1\right\rangle = \left\langle \pi(g)^{-1}\pi(h)^{-1}v_2, v_1\right\rangle = \left\langle \pi(g)^{-1}v_2, v_1\right\rangle = \varphi(g).$$

To compute the action of $\pi(\varphi)$, we proceed indirectly, finding first its matrix coefficients. Let v_i, v_j be basis vectors. Then

$$\begin{aligned}
\langle \pi(\varphi)v_i, v_j\rangle &= \left\langle \int_{g \in G} \varphi(g)\pi(g)v_i d\mu\ ,\ v_j \right\rangle \\
&= \int_{g \in G} \varphi(g)\left\langle \pi(g)v_i, v_j\right\rangle d\mu \\
&= \int_{g \in G} \langle \pi(g)v_i, v_j\rangle\, \overline{\langle \pi(g)v_1, v_2\rangle} d\mu
\end{aligned}$$

We invoke the orthogonality relations for matrix coefficients of π, which are proved in the same manner as for finite groups. According to these relations, the integral is equal to

$$\frac{1}{n}\delta_{i1}\delta_{j2}$$

Thus the matrix for $\pi(\varphi)$ has exactly one nonzero entry, which is in the $(2,1)$ position. We could also define $\varphi' \in \mathcal{H}$ with the roles of v_1 and v_2 reversed:

$$\varphi'(g) = \langle v_1, \pi(g)v_2 \rangle$$

Repeating the above argument, the matrix for φ' has exactly one nonzero entry, which is in the $(1,2)$ position. But these two matrices do not commute, so $\mathcal{H}$ is not commutative:

$$\pi(\varphi * \varphi') = \pi(\varphi)\pi(\varphi') \neq \pi(\varphi')\pi(\varphi) = \pi(\varphi' * \varphi)$$
$$\Rightarrow \varphi * \varphi' \neq \varphi' * \varphi$$

□

3 The Gelfand-Graev Representation

Let $\mathbb{F}_q$ be a finite field, $n \in \mathbb{N}$, $G = GL_n(\mathbb{F}_q)$, and let N be the subgroup of upper triangular matrices with 1 on the diagonal. With ψ an injective complex linear character of $\mathbb{F}_q$, define a one-dimensional representation $\pi\colon N \to \mathbb{C}^\times$ by

$$\pi(a) = \psi(a_{12} + a_{23} + \cdots + a_{n-1,n}), \quad a = (a_{ij}) \in N.$$

The **Gelfand-Graev representation** i_N^G is important in the representation theory of G because it contains most irreducible representations of G. The goal of this section is to show that it is multiplicity-free. We will make use of the Bruhat decomposition, which in the case of GL_n has an easy, direct proof.

Lemma 7 (Bruhat decomposition). *One can decompose $GL_n(\mathbb{F}_q)$ as a disjoint union of double cosets with representatives of a certain form, as follows.*

1. *(The original Bruhat decomposition)*

$$GL_n(\mathbb{F}_q) = \coprod_{w \in W} BwB,$$

   where B is the Borel subgroup of $GL_n(\mathbb{F}_q)$, containing all upper triangular matrices, and W is the group of $n \times n$ permutation matrices.

2. *(A slight modification that will be more convenient for us)*

$$GL_n(\mathbb{F}_q) = \coprod_{m \in M} NmN,$$

   where N is as above and M consists of monomial matrices, i.e. matrices with a single nonzero entry in each row and column.

Proof. (Sketch)

1. At first, ignore the issue of whether the union is disjoint. We proceed by strong induction on n; when $n = 1$, $B = G$ so the result is trivial.

   Suppose that we are given a matrix $g \in GL_n(\mathbb{F}_q)$, $n > 1$. It suffices to find a permutation matrix in BgB. If $g_{n1} \neq 0$ then by multiplying g by appropriate elements of B on the left and right we can erase all the entries in the first column and last row of g except g_{n1} itself, which we can normalize to 1. Apply the induction hypothesis to the matrix obtained by removing the first column and last row of the result. This produces an $(n-1)\times(n-1)$ permutation matrix, and by reinserting the first column and last row we reach the desired form.

If $g_{n1}=0$ then it is trickier. Let $g_{i1}\neq 0$ and $g_{nj}\neq 0$ where i is chosen as large as possible and j as small as possible. By multiplying by elements of B on the right and left, we can clear the first and jth columns and the ith and last rows, except for the entries g_{i1} and g_{nj}, which we can normalize to 1. We can then apply the induction hypothesis to the matrix obtained by removing these rows and columns, to create a permutation matrix. This completes the induction.

We now have to check that the double cosets BwB are disjoint. Let $w_1, w_2 \in W$ be in the same double coset; it follows that

$$w_1 b w_2^{-1} \in B$$

for some $b \in B$. Since $w_1 b w_2^{-1}$ is obtained by permuting rows and columns of b, if we change some of the nonzero entries of b, perhaps making them zero, the result will still be upper triangular. Let us replace b with the identity matrix; then we have

$$\begin{aligned} & w_1 w_2^{-1} \in B \\ \Rightarrow\ & w_1 w_2^{-1} \in W \cap B = \{I_n\} \\ \Rightarrow\ & w_1 = w_2, \end{aligned}$$

as desired.

2. Let $D < G$ be the group of diagonal matrices. The result, ignoring disjointness of the union, follows from the Bruhat decomposition because $B = DN = ND$ and $M = DW = WD$. That the union is disjoint can be deduced as before.

□

Theorem 3. *With N, G, and π as above, the Gelfand-Graev representation $i_N^G\pi$ of $GL_n(\mathbb{F}_q)$ is multiplicity-free.*

Proof. For each double coset NgN, either we will show that $\mathcal{H}$ vanishes on NgN, or we will construct a representative of NgN of a special form. Then we will exhibit an involution of G that preserves these representatives, and will conclude that $\mathcal{H}$ is commutative.

Fix $g \in G$, and consider the double coset NgN. By the lemma, $NgN = NmN$ for some monomial matrix m. Assume that $\mathcal{H}$ does not vanish on NmN, i.e. there is some $\varphi\colon G \to \mathbb{C}$, $\varphi \in \mathcal{H}$, and some $h \in NmN$ such that $\varphi(h) \neq 0$.

The monomial matrix m has a single nonzero entry in each row. We first show that if m_{ij} and $m_{i+1,k}$ are both nonzero entries of m, then $k \leq j+1$. Equivalently, m has the form

$$m = \begin{pmatrix} & & & D_1 \\ & & D_2 & \\ & \iddots & & \\ D_p & & & \end{pmatrix}$$

where D_1, D_2, …, D_p are diagonal matrices.

The proof is by contradiction: assume that $m_{ij}\neq 0$ and $m_{i+1,k}\neq 0$, with $k > j+1$, and define $x, y \in N$ by

$$\begin{aligned} x &= I_n + m_{ij} e_{i,i+1}, \\ y &= I_n + m_{i+1,k} e_{jk}. \end{aligned}$$

(e_{jk} refers to a matrix with a single 1 at position (j,k).) One can verify the following relations:

$$\begin{aligned} xm = m + m_{ij} m_{i+1,k} e_{ik} &= my, \\ \pi(x) = \psi(m_{ij}) &\neq 1, \\ \pi(y) = \psi(0) &= 1. \end{aligned}$$

Any function $\varphi\colon G \to \mathbb{C}$ in the Hecke algebra $\mathcal{H}(G,\pi)$ must satisfy the compatibility property, leading to

$$\pi(x)\varphi(m) = \varphi(xm) = \varphi(my) = \varphi(m)\pi(y)$$

$$\begin{aligned} \Rightarrow (\pi(x) - \pi(y))\varphi(m) &= 0 \\ \Rightarrow \varphi(m) &= 0 \\ \Rightarrow \varphi(NmN) &= 0. \end{aligned}$$

Thus, unless m is of the desired form, every function in $\mathcal{H}$ will vanish on the double coset NmN.

Next, we show that D_t is a scalar matrix, $1 \le t \le p$, by a similar approach. It suffices to show that if m_{ij} and $m_{i+1,j+1}$ are nonzero then they are equal. Let x and y be as above with $k = j+1$; we have as before

$$\begin{aligned} xm &= my, \\ \pi(x) &= \psi(m_{ij}), \\ \pi(y) &= \psi(m_{i+1,j+1}), \\ (\pi(x) - \pi(y))\varphi(m) &= 0. \end{aligned}$$

Since φ does not vanish on NmN, it cannot vanish on m, so

$$\begin{aligned} \psi(m_{i,j}) &= \psi(m_{i+1,j+1}) \\ \Rightarrow m_{i,j} &= m_{i+1,j+1}, \end{aligned}$$

since ψ is injective.

Now consider the involution $\iota\colon G \to G$ taking a matrix g to the matrix obtained by reflecting g about its back-diagonal. Explicitly, $\iota(g) = w(g')w$ where w is the $n \times n$ back-identity matrix and g' denotes the transpose of g. Observe that ι fixes matrices m of the block form described above, where each D_t is a scalar matrix. Since ι also leaves N and π invariant, we can apply Lemma 4 to construct $\tilde{\iota}\colon \mathcal{H} \to \mathcal{H}$. We claim that $\tilde{\iota}(\varphi) = \varphi$ for every $\varphi \in \mathcal{H}$. Indeed, φ is determined by its values on the representatives m we have found, and $\tilde{\iota}(\varphi)$ has the same values on these elements. Hence $\tilde{\iota}$ reduces to the identity map, so that $\mathcal{H}$ is commutative. The result now follows from Lemma 3. □

4 $GL_n(\mathbb{F}_{q^2})$ and $GL_n(\mathbb{F}_q)$

Let q be a prime power, n a positive integer, and $G = GL_n(\mathbb{F}_{q^2})$. Let $F\colon G \to G$ be the **Frobenius automorphism** of G, taking a matrix $a = (a_{ij})$ to $\bar{a} = F(a) = (a_{ij}^q)$. We will use both the bar and F notations, but remember that in the algebraic closure of $\mathbb{F}_q$ it is not true that $\overline{\overline{a}} = a$! Let $F^*\colon G \to G$ denote the **twisted Frobenius automorphism**, given by

$$F^*(a) = (\overline{a}^{-1})',$$

the transpose-inverse of $F(a)$. Consider the sets

$$\begin{aligned} R &= \{r \in G \mid F(r) = r\} = GL_n(\mathbb{F}_q), \\ U &= \{u \in G \mid F^*(u) = u\} = U_n(\mathbb{F}_{q^2}), \\ H &= \{h \in G \mid F(h) = h'\}, \end{aligned}$$

Pretending for a moment that $G = GL_n(\mathbb{C})$ and F is complex conjugation, R would consist of matrices with real entries, U of unitary matrices, and H of Hermitian matrices. We will show that U is a Gelfand subgroup of G. Remarkably, each of our results has an analogue where, roughly, F is replaced by F^*—for example, (G,R) is also a Gelfand pair—but we will not explore this duality.

We rely on an extension of a fundamental theorem of Lang that is often used to study finite groups of Lie type. Let k be the algebraic closure of $\mathbb{F}_q$, $M_n(k)$ be the vector space of $n \times n$ matrices over k, and

$K = GL_n(k)$. If $p_1, \ldots, p_m \colon M_n(k) \to k$ are polynomials in the matrix entries, then we call their set of common roots $S = \{x \in M_n(k) \mid p_1(x) = \cdots = p_m(x) = 0\}$ a **closed set** (in the Zariski topology). The set of invertible matrices in such a closed set, $A = S \cap K$, might happen to form a subgroup of $GL_n(k)$; in this situation we call A a **linear algebraic group**. A inherits the Zariski topology from $M_n(k)$, so that we can ask whether A is connected.

Theorem 4 (Steinberg's Extension of Lang's Theorem). *If A is a connected linear algebraic group and $F \colon A \to A$ is an endomorphism of algebraic groups with finitely many fixed points, then the function $\zeta \colon A \to A$,*

$$\zeta(x) = xF(x)^{-1},$$

is surjective.

Each of F and F^* has finitely many fixed points, and we will use Lang's Theorem in both forms. We begin our analysis of U with the following lemma.

Lemma 8. *Two elements $x,y \in G$ lie in the same U-U double coset if and only if $\overline{x}'x$ and $\overline{y}'y$ are conjugate by an element of U.*

Proof. For one direction, suppose that $y \in UxU$. Thus for some $u, v \in U$,

$$\begin{aligned} y &= vxu \\ \Rightarrow \overline{y}' &= \overline{u}'\overline{x}'\overline{v}' \\ &= u^{-1}\overline{x}'v^{-1} \end{aligned}$$

Combining these two equations, we obtain $\overline{y}'y = u^{-1}\overline{x}'xu$, which expresses the desired result, that $\overline{y}'y$ and $\overline{x}'x$ are conjugate by $u \in U$.

Now assume as hypothesis that $\overline{y}'y = u^{-1}\overline{x}'xu$. Then, rearranging and using $u^{-1} = \overline{u}'$,

$$\begin{aligned} 1 &= (\overline{xuy^{-1}})'xuy^{-1} \\ \Leftrightarrow xuy^{-1} &\in U \\ \Rightarrow x &\in UyU, \end{aligned}$$

as desired. ∎

Thus to each U-U double coset UxU we can associate a matrix $\overline{x}'x \in H$, which is unique up to conjugacy by elements of U. We will see shortly that this uniqueness can be strenghened to conjugacy in G.

Lemma 9. *Hermitian matrices $h_1, h_2 \in H$ are conjugate in G if and only if they are conjugate by an element of U.*

Proof. One direction is trivial. For the other direction, suppose that $h_1 = g^{-1}h_2g$ for some $g \in G$. Using Hermiticity of h_1 and h_2, it follows that

$$\begin{aligned} h_1 &= \overline{g}'h_2\overline{g}'^{-1} \\ \Rightarrow h_2 &= \overline{g}'^{-1}h_1\overline{g}' \\ &= (g\overline{g}')^{-1}h_2g\overline{g}' \\ \Rightarrow g\overline{g}' &\in C_G(h_2) \subset C_K(h_2) \end{aligned}$$

The centralizer $C_K(h_2)$ is known to be a connected linear algebraic group for any $h_2 \in K$. We will use F^* as our Frobenius map, so we should verify that F^* restricts to $C_K(h_2)$. Indeed, if $x \in C_K(h_2)$ then

$$\begin{aligned} xh_2 &= h_2x \\ \Leftrightarrow F^*(x)\,F^*(h_2) &= F^*(h_2)\,F^*(x) \\ \Leftrightarrow F^*(x)h_2^{-1} &= h_2^{-1}\,F^*(x) \\ \Leftrightarrow h_2\,F^*(x) &= F^*(x)h_2 \\ \Leftrightarrow F^*(x) &\in C_K(h_2) \end{aligned}$$

Thus by Lang's theorem, the map $\zeta(x) = x\,F^*(x^{-1})$ of $C_K(h_2)$ into itself is surjective. In particular, there is a preimage $c \in C_K(h_2)$ of $g\overline{g'}$, so that

$$\begin{aligned} c\,F^*(c^{-1}) &= g\overline{g'} \\ &= g\,F^*(g^{-1}) \\ \Rightarrow c^{-1}g &= F^*(c^{-1}g) \\ \Leftrightarrow c^{-1}g &\in U \end{aligned}$$

We would like to conclude that $c^{-1}g \in U$ by definition, but we need to know first that $c^{-1}g \in G$. However,

$$F^2(c^{-1}g) = F^{*2}(c^{-1}g) = c^{-1}g,$$

so that the entries of $c^{-1}g$ are fixed by F^2, hence they are in $\mathbb{F}_{q^2}$, and $c^{-1}g \in G$. Now, since c commutes with h_2, we obtain the desired relation,

$$\begin{aligned} h_1 &= g^{-1}h_2g \\ &= g^{-1}(ch_2c^{-1})g \\ &= (c^{-1}g)^{-1}h_2(c^{-1}g) \end{aligned}$$

Therefore $c^{-1}g$ is an element of U that conjugates h_2 to h_1. □

Theorem 5. *$U = U_n(\mathbb{F}_q)$ is a Gelfand subgroup of $G = GL_n(\mathbb{F}_{q^2})$.*

Proof. Let $\iota\colon G \to G$ denote the map

$$\iota(x) = \overline{x}'$$

Since ι is a composition of the Frobenius automorphism and the transpose involution, and since $\iota^2 = \mathrm{id}$, we conclude that ι is an involution of G. By Theorem 1, it remains to show that ι preserves U-U double cosets. Let UxU be an arbitrary double coset, $x \in G$. The Hermitian matrices corresponding to the double cosets UxU and $U\iota(x)U$ are respectively $\overline{x}'x = \iota(x)x$ and $x\iota(x)$ which are conjugate by $x \in G$. By Lemma 9, $x\iota(x)$ and $\iota(x)x$ are also conjugate by an element of U, thus Lemma 8 implies that $UxU = U\iota(x)U$. We have arrived at the conditions of Theorem 1. □

Knowing that the irreducible constituents of $i_U^G 1$ are distinct, we may now wish to describe them, or at least count them. By Theorem 1, they are equal in number to the U-U double cosets in G. Lemma 8 describes an injection of the latter into the set of U-conjugacy classes of H, taking the double coset UxU to the conjugacy class of $\overline{x}'x$. The following lemma shows that this map is onto.

Lemma 10. *For each $h \in H$, there is an $x \in G$ such that $\overline{x}'x = h$.*

Proof. Lang's Theorem applied to K and F^* immediately gives us $y \in K$ satisfying the related condition $y\overline{y}' = h$. We have

$$\begin{aligned} yF(y)' &= h \\ \Rightarrow F^2(y)F(y)' &= F(h)' = h \end{aligned}$$

Combining these equations gives $y = F^2(y)$, so that $y \in G$. Now $x = \overline{y}' \in G$ is the desired element because $\overline{x}x = y\overline{y} = h$. □

Next is another technical lemma relating conjugacy in different subgroups of K.

Lemma 11.

1. *Two matrices $r_1, r_1 \in R$ are conjugate in R if and only if they are conjugate in K.*
2. *Two matrices $g_1, g_2 \in G$ are conjugate in G if and only if they are conjugate in K.*

Proof. The two parts are really the same statement for over different fields, because $R = GL_n(\mathbb{F}_q)$ while $G = GL_n(\mathbb{F}_{q^2})$; we should only prove part 1, and only the reverse direction. The proof is very similar to that of Lemma 9, so we will skip the computations. Let $r_1, r_2 \in R$ satisfy $r_1 = x^{-1} r_2 x$ for some $x \in K$. Note that $x\overline{x}^{-1} \in C_K(r_2)$, and that the centralizer $C_K(r_2)$ is a connected linear algebraic group on which F acts. Applying Lang's Theorem to $C_K(r_2)$ and F we find $c \in C_K(r_2)$ such that

$$c\overline{c}^{-1} = x\overline{x}^{-1} \in C_K(r_2).$$

Then $c^{-1}x \in R$ performs the required conjugation,

$$r_1 = (c^{-1}x)^{-1} r_2 c^{-1} x.$$

□

Using Lemmas 8, 9, and 10, we can relate the number of U-U double cosets to the number of G-conjugacy classes of H. In fact, these conjugacy classes of H have a simpler interpretation in terms of R.

Lemma 12. *A conjugacy class of G contains an element of H if and only if it contains an element of R. Thus, the number of U-U double cosets in G is equal to the number of conjugacy classes of R.*

Proof. For the first claim, we must show that each element of H is conjugate to some element of R and vice versa.

Let $h \in H$. Any matrix is conjugate in K to its transpose–this is not hard to verify if the matrix is in Jordan canonical form–so that there is some $y \in K$ such that

$$y^{-1}hy = h' = \overline{h}.$$

We apply Lang's theorem directly to K and F: it states that $x\overline{x}^{-1} = y$ should have a solution $x \in K$. Let $r = x^{-1}hx$; then

$$\begin{aligned} \overline{r} &= \overline{x}^{-1} h' \overline{x} \\ &= (x^{-}1y)(y^{-1}hy)(y^{-}1x) \\ &= x^{-}1hx \\ &= r \\ \Rightarrow r &\in R \end{aligned}$$

Finally, by Lemma 11, r and h are conjugate not only in K but also in G.

Now suppose we are given $r \in R$. Again, let $y \in K$ satisfy

$$y^{-1}ry = r'.$$

This time we apply Lang's theorem to K and F^*, to obtain $x \in K$ such that $x\overline{x}' = y$. As before, $h = x^{-1}rx$ is Hermitian and is conjugate to r in G.

We know that U-U double cosets of G are in bijective correspondence with the conjugacy classes of G that contain elements of H. As we have just seen, these are also the classes that contain elements of R. Each such class is an entire conjugacy class of R, by Lemma 11, hence the last sentence in the statement of the lemma follows. □

F permutes the irreducible representations of G in a natural way: If (ρ, W) is an irreducible representation of G, then $(F\rho, W) = (\overline{\rho}, W)$ is the irreducible representation

$$\overline{\rho}(g)w = \rho(\overline{g})w,$$

for $w \in W$. It may happen that $\overline{\rho} \simeq \rho$; we will show that this occurs precisely when ρ is in the decomposition of $i_U^G 1$. First, we state a result from character theory:

Lemma 13 (Brauer Permutation Lemma). *Let A be a group which acts on the set of irreducible characters of G and on the set of conjugacy classes of G. Assume that $\chi(g) = (a \cdot \chi)(a \cdot g)$ for each $a \in A$, $g \in G$, and irreducible character χ. Then for each $a \in A$, the number of fixed irreducible characters of G is equal to the number of fixed classes.*

We will apply this to the cyclic group $A = \{1, a\}$ of order 2, where a acts by the Frobenius map F. We have defined the action of F on representations so that it obeys the consistency condition in this lemma. Now it is easy to see that an irreducible character χ_ρ of G is fixed by a if and only if $\rho \simeq \overline{\rho}$, but determining which conjugacy classes are fixed is more subtle.

Lemma 14. *A conjugacy class of G is fixed by F if and only if it contains an element of R.*

Proof. If $r \in R$ and $g = t^{-1}rt$ is conjugate to r, then $\overline{g}$ is also conjugate to $\overline{r} = r$ because $\overline{g} = \overline{t}^{-1} r \overline{t}$. This remark proves one direction of the claim. For the other direction, suppose that $g \in G$ is conjugate to $\overline{g}$, say by $y^{-1}gy = \overline{g}$. By Lang's theorem applied to K and F, there is an element $x \in K$ such that $x\overline{x}^{-}1 = y$. Let $r = x^{-1}gx$; we proceed as in the first part of Lemma 12 to show that $r \in R$ and that r and g lie in the same conjugacy class of G. □

Now Brauer's Permutation Lemma tells us that the number of irreducible representations fixed by F is equal to the number of conjugacy classes of G containing elements of R, that is, by Lemma 11, the number of conjugacy classes of R.

Theorem 6. *The irreducible subrepresentations of $i_U^G 1$ are precisely the irreducible representations ρ of G such that $\rho \simeq \overline{\rho}$.*

Proof. We have shown in the previous discussion and Lemma 12 that the two sets in the statement have the same cardinality, namely, the number of conjugacy classes of R. Thus it is sufficient to prove that each irreducible subrepresentation of $i_U^G 1$ is equivalent to its image under F.

Let (ρ, V) be an irreducible subrepresentation of $i_U^G 1$. Recall from Exercise 6 on page 33 of the course notes that ρ induces an irreducible representation of $\mathcal{H} = \mathcal{H}(G, 1_U)$,

$$f \cdot v = \frac{1}{|G|} \sum_{g \in G} f(g)\rho(g)v,$$

for vectors v in the one-dimensional subspace V^U of U-fixed vectors. Replacing the index g in the sum with $\overline{g}$, we find that

$$\begin{aligned} f \cdot v &= \frac{1}{|G|} \sum_{g \in G} f(\overline{g})\rho(\overline{g})v \\ &= \frac{1}{|G|} \sum_{g \in G} f(g)\overline{\rho}(g)v \end{aligned}$$

To justify the last equality, we claim that g and $\overline{g}$ lie in the same U-U double coset; because $f \in \mathcal{H}$ is constant on such double cosets, this implies $f(g) = f(\overline{g})$. According to Lemmas 8, 9, and 11, it is sufficient to show that $\overline{g'}g$ and $g'\overline{g}$ are conjugate in K, which is true because they are transposes of each other.

Comparing the last two equations, we find that ρ and $\overline{\rho}$ must induce the same representation of $\mathcal{H}$. Strictly, we should check that V^U is the set of U-fixed vectors in $\overline{\rho}$, but this is straightforward because $U = F(U)$. It follows that ρ and $\overline{\rho}$ are equivalent representations of G. □

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