

A Parity Subtraction Game

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In memory of Robert Barrington Leigh

In the *Olympiad Corner* No. 222 of *CRUX with MAYHEM*, 28, no. 4 (May, 2002), a selection of problems from the St. Petersburg Mathematical Olympiads is given by Oleg Ivrii and Robert Barrington Leigh. The third one [2, p. 289, Problem 3 (1965)] is

A game starts with a heap of 25 beans. Two players alternately remove 1, 2, or 3 of them. When all the beans have been taken, the winner is the player who has an even number of beans. Assuming perfect play, does the first player or the second have a sure win?

The *Olympiad Corner* editor recently received a request for a solution. The problem is from a list of supplementary problems; it may not have been used, and no solution is given in the book.

This game differs from the usual kind of take-away game in that it is not impartial. After a move has been made, even if you were not watching, you can tell which player has moved by noticing who has added to his or her collection of beans. Thus, we cannot use the Sprague-Grundy Theory; that is, we cannot calculate nim-values [1, Chap. 2]. On the other hand, it is not always a last-player-winning game; there is a mixture of normal and misère (last-player-losing) play [1, Chap. 13], which means that we cannot use the Conway Theory [1, Chap. 1] either.

We use a rather brute-force method, in effect drawing the whole game tree, though we save a good deal of space by identifying nodes in our Figure 1.

Here is a solution for the game, played with any odd number of beans.

For heaps of $8k + 3$, $8k + 5$, or $8k + 7$ beans, the first player wins; for heaps of $8k + 1$ beans, the second player wins. Hence, with 25 beans, the second player can win.

To see that this is the case, we represent positions in the game by $b(f, s)$, where b is the number of beans remaining in the heap, and f and s are the total numbers of beans already collected by the first (next) and second (previous) players, respectively.

We will use d and e for arbitrary odd and even numbers, respectively. The opening position is of shape $d(0, 0)$, and subsequent positions all satisfy $b + f + s = d$. Notice that, when a move is made, the roles of first (next) and second (previous) player are interchanged. When the next player takes t beans from $b(f, s)$, the position becomes $b-t(s, f+t)$. There is need for someone to devise a more perspicuous notation!

Detailed Solution

A position in such a game is, with best play, either a win for the first (next) player, or the second (previous) player. These are often labelled \mathcal{N} -positions and \mathcal{P} -positions, respectively.

Solution. In the game described above,

1. the \mathcal{P} -positions are $8k+1 (e, e)$ and $8k+5 (d, d)$;
2. the \mathcal{N} -positions are $8k+7 (d, d)$, $8k+7 (e, e)$, $8k+6 (d, e)$, $8k+6 (e, d)$, $8k+5 (e, e)$, $8k+3 (e, e)$, $8k+3 (d, d)$, $8k+2 (d, e)$, $8k+2 (e, d)$ and $8k+1 (d, d)$.
3. That leaves $8k+4 (d, e)$ and $8k+4 (e, d)$, which are wins for d , whoever starts, and $8k (d, e)$ and $8k (e, d)$, which are wins for e , whoever starts. That is, $8k+4 (d, e)$ and $8k (e, d)$ are \mathcal{N} -positions, while $8k+4 (e, d)$ and $8k (d, e)$ are \mathcal{P} -positions.

Proof: We use induction. To start the induction, we consider $k = 0$.

- $0 (d, e)$ is a win for e by definition, and hence, $1 (e, e)$ will be a loss, and $1 (d, d)$ a win, for the next player.
- $2 (d, e)$ is a win for the next player, if he goes to $1 (e, e)$, as is $2 (e, d)$ if she takes both beans, going to $0 (d, e)$.
- If there are just 3 beans remaining, the first player takes them all if he is d , but only 2 of them if she is e .
- But $4 (d, e)$ and $4 (e, d)$ are both wins for d . If d starts, he goes to $1 (e, e)$, while the only possible moves for e are $3 (d, d)$, $2 (d, e)$, or $1 (d, d)$, which are all next-player wins.
- Thus, from $5 (e, e)$, the next player will go to $4 (e, d)$. But from $5 (d, d)$, the next player must go to $4 (d, e)$, $3 (d, d)$, or $2 (d, e)$, which are next-player wins.
- $6 (d, e)$ and $6 (e, d)$ are \mathcal{N} -positions; d goes to $4 (e, d)$ or e goes to $5 (d, d)$.
- From $7 (e, e)$, the next player goes to $4 (e, d)$, where we have seen that d wins, while from $7 (d, d)$ the next player can go to the \mathcal{P} -position $5 (d, d)$.
- From $8 (e, d)$, e wins by going to $5 (d, d)$ while from $8 (d, e)$, d must play to $7 (e, e)$, $6 (e, d)$, or $5 (e, e)$ and the next player then wins by going to $4 (e, d)$, $5 (d, d)$, or $4 (e, d)$, respectively.

In order to get the induction off the ground, we need to go a bit further, with three cases of $k = 1$, since the players may remove up to three beans. From $9 (d, d)$, the next player wins by going to $8 (d, e)$, while $9 (e, e)$ is a \mathcal{P} -position, the second player able to go to $5 (d, d)$ if the first player takes 1 or 3, or to $4 (e, d)$ if the first player takes 2.

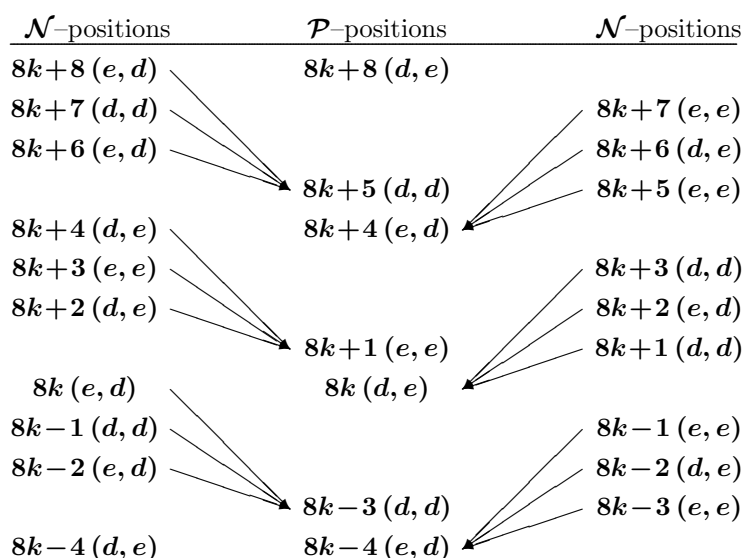


Figure 1: Condensed game tree

From $10 (d, e)$, d wins with $9 (e, e)$ and from $10 (e, d)$, e wins with $8 (d, e)$. From here on, copy the strategy from 8 beans back; for example, from $11 (d, d)$ move to $8 (d, e)$ and from $11 (e, e)$ to $9 (e, e)$.

Check the detailed solution listed above against Figure 1, which is periodic in the sense that it repeats itself every 8 rows. For example,

the strategy for $8k+3$ $8k+4$ $8k+5$ $8k+6$ $8k+7$
 is the same as for $8k-5$ $8k-4$ $8k-3$ $8k-2$ $8k-1$

respectively. The arrows in Figure 1 are the only winning moves. There is an arrow from every \mathcal{N} -position to a \mathcal{P} -position. All other legal moves, of which there are three from each \mathcal{P} -position and two non-winning moves from each \mathcal{N} -position, lead to \mathcal{N} -positions; they are not shown in the figure.

Can anyone supply a general theory for such “four outcome” games?

References

- [1] Elwyn Berlekamp, John Conway, and Richard Guy, *Winning Ways for Your Mathematical Plays*, 2nd edition, A.K. Peters, Natick, MA, 2001–2004.
- [2] D.V. Fomin, *St. Petersburg Mathematical Olympiads, 1961–1993*, Politechnika, St. Petersburg, 1994.

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