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Character values for $\mathrm{GL}(2, \mathbb{Z}/p^\ell\mathbb{Z})$ ☆

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ABSTRACT

We give the values of the irreducible complex characters of the group $\mathrm{GL}(2, \mathbb{Z}/p^\ell\mathbb{Z})$ where ℓ is an integer > 1.

1. Introduction

We determine the values of the irreducible complex characters of the general linear group $\mathrm{GL}(2, \mathbb{Z}/p^\ell\mathbb{Z})$ over the ring $\mathbb{Z}/p^\ell\mathbb{Z}$, where ℓ is an integer > 1 and p is an odd prime.

The irreducible representations were constructed by Nobs [3], using the Weil representation. However Nobs did not consider the problem of finding the character values.

Our methods are quite different than those of Nobs; we rely on Clifford theory. Our methods of constructing the irreducible characters are somewhat similar to those of Kutzko [2], who was interested in representations of $\mathrm{GL}(2, F)$ where F is a p-adic field. Indeed, one of the reasons for our interest in this problem is that smooth, irreducible super-cuspidal representations of $\mathrm{GL}(2, F)$ are induced from those of $\mathrm{GL}(2, \mathcal{O})$ where $\mathcal{O}$ is the ring of integers of F, and these in turn arise from representations of $\mathrm{GL}(2, k)$ where k is a finite factor ring of $\mathcal{O}$.

We deal with non-linear irreducible characters which do not come from $\mathrm{GL}(2, \mathbb{Z}/p^{\ell-1}\mathbb{Z})$. These have three possible degrees. There are characters coming from parabolic induction, which in this case means induced from a Borel subgroup; these have degree $(p+1)p^{\ell-1}$. There are two other families of characters, of degrees $(p-1)p^{\ell-1}$ and $(p^2-1)p^{\ell-2}$. The irreducible characters are induced from a character ψ of the stabilizer of a degree-one character defined on a suitable congruence subgroup K_i; this stabilizer has the form $K_j S$ where S is an analogue of a maximal torus. The theory is easier

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if ℓ is even, when $i = j = \ell/2$ and ψ has degree 1; however some of the character values, such as formula (5.4) turn out to have the same form in the even and odd cases.

For the last family, Kloosterman sums appear in the computation of the character values. In [4] and [5], Takahashi also encountered Kloosterman sums, while considering characters of infinite-dimensional representations of $\mathrm{GL}(\ell, F)$. It is not clear what, if any, is the relation between our characters and those in [4] and [5].

Our methods could be used to find the character values for $\mathrm{GL}(2, R/P^\ell)$ where R is the ring of integers of a local or global field, and P is a prime ideal of R. It is easier for us to count the number of irreducible characters of each degree in the case that $R = \mathbb{Z}$.

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2. Conjugacy classes

Let R be a finite commutative principal local ring of odd characteristic. This means that there is a nilpotent element $\pi \in R$ and a positive integer ℓ such that $\pi^\ell = 0$ and every nonzero $x \in R$ can be written as

$$x = u\pi^k$$

for some $u \in R^\times$ and a unique k, $0 \leqslant k < l$. In particular, $R = R^\times \dot\cup \pi R$, so that $R/\pi R = \mathbb{F}_q$, where $q = |R/\pi R|$.

Let

$$M = \{2 \times 2 \text{ matrices over } R\}, \qquad M_0 = \big\{A \in M\colon \operatorname{tr}(A) = 0\big\},$$
$$G = \mathrm{GL}(2, R) = M^\times = \big\{A \in M\colon \det(A) \in R^\times\big\}.$$

Let I denote the identity matrix. Now G acts on M by conjugation, preserving both trace and determinant. It follows that the conjugacy class of an element g in G is equal to the orbit of the same matrix $g \in M$ under this action. Moreover, the action restricts nontrivially to M_0 and trivially to $\{\alpha I\colon\ \alpha \in R\}$. Since these two subgroups generate M^+ additively (provided that R has odd characteristic), it is sufficient to describe the orbits in M_0.

We identify some invariant subgroups of M_0. Let $L_i = \pi^i M_0$, $0 \leqslant i \leqslant l$, the subset of matrices all of whose entries are multiples of π^i. This is invariant under the action of G because the constant π^i factors through the conjugation. We have $\{0\} \subset L_l \subset L_{l-1} \subset \cdots \subset L_0 = M_0$.

Let $A \in M_0$ be a matrix that is not in L_1. We will find a canonical representative for the similarity class (orbit) of A. Form the 2×2 matrix B over $\mathbb{F}_q = R/\pi R$ by reducing the entries of A modulo πR. Then B has trace 0 but is not the zero matrix, hence it is not a multiple of the identity matrix. We therefore know that B is conjugate by some element of $\mathrm{GL}(2, \mathbb{F}_q)$ to the matrix $\left(\begin{smallmatrix} & -\det(B) \\ 1 & \end{smallmatrix}\right)$. It follows that A is conjugate by some element of G to the matrix

$$A' = \begin{pmatrix} \pi\alpha & \beta \\ 1 + \pi\gamma & -\pi\alpha \end{pmatrix}$$

for some $\alpha, \beta, \gamma \in R$, $\beta \equiv -\det(B) \pmod{\pi R}$. Since

$$\begin{pmatrix} 1 & \pi\alpha \\ & 1 + \pi\gamma \end{pmatrix}^{-1} A' \begin{pmatrix} 1 & \pi\alpha \\ & 1 + \pi\gamma \end{pmatrix} = \begin{pmatrix} & \pi^2\alpha^2 + (1 + \pi\gamma)\beta \\ 1 & \end{pmatrix} = \begin{pmatrix} & -\det(A') \\ 1 & \end{pmatrix},$$

we have shown that the orbit of A contains a unique representative of the form $\begin{pmatrix} & \beta \\ 1 & \end{pmatrix}$. The set of all such matrices, with $\beta \in R$, contains one representative from each orbit in $M_0 \setminus L_1$.

Now consider any matrix $C \in L_i \setminus L_{i+1}$, $0 \leqslant i < l$. Thus $C = \pi^i A$ for some matrix $A \in M_0 \setminus L_1$. We will reduce the problem of finding a representative for the orbit of C to the special case we have already solved, but over a different ring. Let $R_i = R/\pi^{l-i}R$. There is an additive isomorphism $\theta : R_i \to \pi^i R$ given by $\theta(x) = \pi^i x$. Write θ also for the corresponding map of 2×2 matrices, and use bars to denote reduction modulo $\pi^{l-i}R$. Hence, for $g \in G$,

$$gCg^{-1} = g(\pi^i A)g^{-1} = \pi^i gAg^{-1} = \theta(\bar{g}\bar{A}\bar{g}^{-1}).$$

That is, the orbit of $C = \theta(\bar{A})$ under the action of $\mathrm{GL}(2, R)$ is $\theta(O)$ where O is the orbit of $\bar{A}$ under the action of $\mathrm{GL}(2, R_i)$. We know the orbit representatives for this action, because it is the special case considered before. Thus holding i fixed, C must be in the orbit of exactly one of

$$\begin{pmatrix} & \beta \\ \pi^i & \end{pmatrix}_{\beta \in \pi^i R.}$$

There is another special case, $i = l$, but it includes only the 0 matrix.

As remarked before, knowledge of the similarity classes of M_0 implies knowledge of the similarity classes of M and the conjugacy classes of G. The following is a set of similarity class representatives for M: the representatives of M, plus arbitrary multiples of the identity:

$$\begin{pmatrix} \alpha & \beta \\ \pi^i & \alpha \end{pmatrix}_{0 \leqslant i \leqslant l, \alpha \in R, \beta \in \pi^i R.}$$

For conjugacy class representatives of G, it suffices to discard singular matrices from the above list. We can make a more useful list at the cost of distinguishing a few cases. First we have the case $i = l$, for which the matrix is a multiple of the identity. Otherwise, fix $i < \ell$, and let $\beta = \theta(\gamma)$, $\gamma \in R_i$.

If γ is the square of a unit in R_i, say $\gamma = \delta^2 \in R_i^\times$, then

$$\begin{pmatrix} \alpha & \beta \\ \pi^i & \alpha \end{pmatrix} \sim \begin{pmatrix} \alpha + \pi^i \delta & \\ & \alpha - \pi^i \delta \end{pmatrix}.$$

Note that δ is only defined up to sign.

Fix a non-square unit ϵ in R. If γ is a non-square unit in R_i, then $\gamma \bar{\epsilon}^{-1}$ is a square, say $\gamma = \delta^2 \bar{\epsilon}$, $\delta \in R_i^\times$. In this case,

$$\begin{pmatrix} \alpha & \beta \\ \pi^i & \alpha \end{pmatrix} \sim \begin{pmatrix} \alpha & \pi^i \delta \epsilon \\ \pi^i \delta & \alpha \end{pmatrix}.$$

Once again, δ is only defined up to sign.

For the remaining case when γ is not a unit, we have $\beta = \pi^{i+1}\beta'$ and therefore

$$\begin{pmatrix} \alpha & \beta \\ \pi^i & \alpha \end{pmatrix} = \begin{pmatrix} \alpha & \pi^{i+1}\beta' \\ \pi^i & \alpha \end{pmatrix}$$

is another class type.

Summary: Conjugacy classes of GL(2, R), where $R = \mathbb{Z}/p^\ell\mathbb{Z}$.

Name of class type	Parameters	Representatives
I_α	$\alpha \in R^\times$	$\begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}$
$B_{i\alpha\beta}$	$0 \leqslant i < \ell,\ \alpha \in R^\times,\ \beta \in R/p^{\ell-i-1}R$	$\begin{pmatrix} \alpha & p^{i+1}\beta \\ p^i & \alpha \end{pmatrix}$
$C_{i\alpha\beta}$	$0 \leqslant i < \ell,\ \alpha \in R,\ \beta \in R^\times,\ \alpha^2 - \epsilon\beta^2 p^{2i} \in R^\times$	$\begin{pmatrix} \alpha & p^i\epsilon\beta \\ p^i\beta & \alpha \end{pmatrix}$
$D_{i\alpha\delta}$	$0 \leqslant i < \ell,\ \alpha, \delta \in R^\times,\ \alpha - \delta \in p^i R^\times$	$\begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix}$

Name	Number of classes if $i = 0$	Number of classes if $i > 0$	Size of class
I_α	–	$(p-1)p^{\ell-1}$	1
$B_{i\alpha\beta}$	$(p-1)p^{2\ell-2}$	$(p-1)p^{2\ell-i-2}$	$(p-1)(p+1)p^{2\ell-2i-2}$
$C_{i\alpha\beta}$	$\frac{1}{2}(p-1)p^{2\ell-1}$	$\frac{1}{2}(p-1)^2 p^{2\ell-i-2}$	$(p-1)p^{2\ell-2i-1}$
$D_{i\alpha\delta}$	$\frac{1}{2}(p-1)(p-2)p^{2\ell-2}$	$\frac{1}{2}(p-1)^2 p^{2\ell-i-2}$	$(p+1)p^{2\ell-2i-1}$

3. The irreducible characters GL($2, \mathbb{Z}/p^\ell\mathbb{Z}$)

In this section, we construct three types of irreducible characters of $\mathrm{GL}(2, \mathbb{Z}/p^\ell\mathbb{Z})$, then count the number to see that we have all of them.

3.1. *Construction of irreducible characters of* $\mathrm{GL}(2, \mathbb{Z}/p^\ell\mathbb{Z})$

We will apply Clifford theory to construct 3 kinds of irreducible characters of $G = \mathrm{GL}(2, \mathbb{Z}/p^\ell\mathbb{Z})$. Let $N \lhd G$ be a normal subgroup of G. For a character ϕ of N, define

$$\phi^g : N \to C, \quad \phi^g(n) = \phi\big(gng^{-1}\big), \quad g \in G,\ n \in N.$$

Then ϕ^g, the conjugate of ϕ, is also a character of N. Let $\mathrm{Irr}(N)$ be the set of irreducible characters of N; denote $T = \mathrm{Stab}_G(\phi) = \{g \in G \mid \phi^g = \phi\}$. Let $\phi \in \mathrm{Irr}(N)$. Clifford theory, e.g. [1, Theorem 6.11], implies that there exists $\psi \in \mathrm{Irr}(T)$ such that $\psi|_N$ is a multiple of ϕ, namely the inner product $(\psi|_N, \phi) \neq 0$, and then the induced character $\mathrm{Ind}_T^G \psi$ is in $\mathrm{Irr}(G)$. Also, the map $\psi \to \mathrm{Ind}_T^G \phi$ is a bijection of $\{\psi \in \mathrm{Irr}(T) \mid (\psi|_N, \phi) \neq 0\}$ onto $\{\chi \in \mathrm{Irr}(G) \mid (\chi|_N, \phi) \neq 0\}$.

Let $p > 2$ be prime, $l \geqslant 2$ be a positive integer, $R = \mathbb{Z}/p^\ell\mathbb{Z}$, $m = \lfloor l/2 \rfloor$, $G = \mathrm{GL}(2, \mathbb{Z}/p^\ell\mathbb{Z})$, and $K_i = \{I + p^i B\colon\ B \in M_{2\times 2}(R)\}$ for $1 \leqslant i < \ell$. Note that $K_i \trianglelefteq G$, for all i, and that K_i is abelian if $i \geqslant l/2$, because $(I + p^i B)(I + p^i C) = I + p^i(B + C)$. Since

$$K_i = \left\{ \begin{pmatrix} 1 + p^i a & p^i b \\ p^i c & 1 + p^i d \end{pmatrix} :\ 0 \leqslant a, b, c, d < p^{\ell-i} \right\} \quad \text{then } |K_i| = p^{4(\ell-i)}.$$

We have the natural homomorphism $\mathrm{GL}(2, Z/p^\ell\mathbb{Z}) \to \mathrm{GL}(2, Z/p\mathbb{Z})$ whose kernel is K_1. It follows that

$$\big|\mathrm{GL}\big(2, Z/p^\ell\mathbb{Z}\big)\big| = p^{4(\ell-1)}\big(p^2 - 1\big)\big(p^2 - p\big) = p^{4\ell-3}\big(p^2 - 1\big)(p - 1). \tag{3.1}$$

Characters with kernel containing K_{l-1} are lifted from the quotient group G/K_{l-1} which is isomorphic to $\mathrm{GL}(2, Z/p^{\ell-1}\mathbb{Z})$; we assume that these are already known inductively.

We first describe the characters of the abelian group K_i, $i \geqslant l/2$. Assign a fixed injective homomorphism

$$\lambda : \mathbb{Z}/p^\ell\mathbb{Z} \to \mathbb{C}^\times.$$

Let A be a 2×2 matrix with entries in $\mathbb{Z}/p^\ell\mathbb{Z}$, and define ϕ_A by

$$\phi_A\big(I + p^i B\big) = \lambda\big(\mathrm{tr}\big(p^i AB\big)\big) \tag{3.2}$$

giving a character on K_i of degree 1.

If A is replaced by the matrix of the form $A + p^{\ell-i}A'$ then $\phi_A = \phi_{A+p^{\ell-i}A'}$, so need only consider those matrices $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with $0 \leqslant a, b, c, d < p^{\ell-i}$. So we can consider A to be a matrix, denoted A_0, over $\mathbb{Z}/p^{\ell-i}\mathbb{Z}$ and then the irreducible characters ϕ_{A_0} of K_i are in one-to-one correspondence with 2×2 matrices A_0 over $\mathbb{Z}/p^{\ell-i}\mathbb{Z}$. By the definition of K_i, we can also treat B as a matrix B_0 over $\mathbb{Z}/p^{\ell-i}\mathbb{Z}$. We can find an injective homomorphism $\lambda_0 : \mathbb{Z}/p^{\ell-i}\mathbb{Z} \to \mathbb{C}^\times$ such that $\lambda(\mathrm{tr}(p^i AB)) = \lambda_0(\mathrm{tr}(A_0 B_0))$. Thus we can also define ϕ_A by

$$\phi_A\big(I + p^i B\big) = \lambda_0\big(\mathrm{tr}(A_0 B_0)\big). \tag{3.3}$$

An element $g \in G$ acts on K_i by conjugation via $(I + p^i B)^g = I + p^i gBg^{-1}$ and g also acts on the characters of K_i. We have the following elementary result.

Lemma 3.1. *For matrices A and A', the characters ϕ_A and $\phi_{A'}$ are conjugate if and only if the matrices A and A' are conjugate.*

Proof.

$$\begin{aligned}
(\phi_A)^g\big(I + p^i B\big) &= \phi_A\big(I + p^i gBg^{-1}\big) \\
&= \lambda\big(\mathrm{tr}\big(p^i AgBg^{-1}\big)\big) \\
&= \lambda\big(\mathrm{tr}\big(p^i g^{-1}AgB\big)\big) \\
&= \phi_{A^{g^{-1}}}\big(I + p^i B\big). \qquad \square
\end{aligned}$$

The stabilizer of ϕ_A is

$$T = \mathrm{Stab}_G(\phi_A) = \big\{g \in G:\ p^i gA = p^i Ag\big\}. \tag{3.4}$$

Clifford's theorem implies that all the characters of G can be obtained by inducing from T to G all possible characters ψ of T that restrict to multiples of ϕ_A on K_i.

We will concentrate on three different types of matrix A, which will give us characters χ of G of three possible degrees. We will then show that we get all possible irreducible characters of G.

3.2. The even case

When $l = 2m$ is even, the group K_m is abelian, and this will allow us to construct characters of G easily. The process is as follows. We will see that T has the form $K_m S$ for some abelian subgroup S of G, depending on the choice of A. Since S is abelian, we can extend the character ϕ_A of K_m to a character ψ of T, and then induce ψ to G. Pictorially, we have

$$\begin{array}{ccccc} K_m & \longrightarrow & T & \longrightarrow & G \\ \phi_A & \xrightarrow{\text{ext}} & \psi & \xrightarrow{\text{ind}} & \chi \end{array}.$$

3.2.1.

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad \phi_A \begin{pmatrix} 1+p^m a & p^m b \\ p^m c & 1+p^m d \end{pmatrix} = \lambda(p^m a).$$

It follows from (3.4) that the stabilizer T of ϕ_A is given by

$$T = \left\{ \begin{pmatrix} a & p^m b \\ p^m c & d \end{pmatrix} : a, d \in (\mathbb{Z}/p^\ell\mathbb{Z})^\times,\ 0 \leqslant b, c < p^{m+1} \right\} = K_m S,$$

$$S = \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} : a, d \in (\mathbb{Z}/p^\ell\mathbb{Z})^\times \right\}, \qquad |T| = (p^\ell - p^{\ell-1})^2 p^\ell = p^{3\ell-2}(p-1)^2. \tag{3.5}$$

Let $\lambda' : (\mathbb{Z}/p^\ell\mathbb{Z})^\times \to \mathbb{C}^\times$ be a multiplicative character such that $\lambda'(1+p^m a) = \lambda(p^m a)$. We can extend ϕ to a character ψ of T by

$$\psi \begin{pmatrix} a & p^m b \\ p^m c & d \end{pmatrix} = \lambda'(a).$$

From (3.1) it follows that the index $|G : T|$, and hence the character degree of $\chi = \operatorname{Ind}\psi$, is $p^{\ell-1}(p+1)$.

3.2.2. $A = \left(\begin{smallmatrix} 0 & \epsilon \\ 1 & 0 \end{smallmatrix}\right)$, where ϵ is a non-square unit. From (3.4),

$$T = \mathrm{GL}(2, \mathbb{Z}/p^\ell\mathbb{Z}) \cap \left\{ \begin{pmatrix} x+p^m a & \epsilon y + p^m b \\ y + p^m c & x + p^m d \end{pmatrix} \right\} = K_m S, \quad S = \mathrm{GL}(2, \mathbb{Z}/p^\ell\mathbb{Z}) \cap \left\{ \begin{pmatrix} x & \epsilon y \\ y & x \end{pmatrix} \right\}.$$

Lemma 3.2. *The group S has order $(p^2-1)p^{2\ell-2}$.*

Proof. Modulo K_1, we have $S/S \cap K_1 \cong S_1$, where $S_1 = \left\{\left(\begin{smallmatrix} x & y\epsilon \\ x & y \end{smallmatrix}\right)\right\} \cap \mathrm{GL}(2, \mathbb{Z}/p\mathbb{Z})$. Since S_1 is isomorphic to the group of units of the field $GF(p^2)$, then $|S_1| = p^2 - 1$, so $|S| = (p^2-1)p^{2\ell-2}$. □

Since S is abelian, we can extend ϕ_A to a character ψ of T; $|T| = |K_m||S|/|K_m \cap S|$. Since

$$K_m \cap S = \left\{ \begin{pmatrix} 1+p^m a & \epsilon p^m b \\ p^m b & 1+p^m a \end{pmatrix} \right\}$$

then $|K_m \cap S| = p^{2m} = p^\ell$. Then

$$|T| = p^{2\ell}(p^2-1)p^{2\ell-2}/p^\ell = p^{3\ell-2}(p^2-1).$$

It follows from (3.1) that the degree of $\chi = |G : T| = p^{\ell-1}(p-1)$.

3.2.3. $A = \begin{pmatrix} 0 & p\beta \\ 1 & 0 \end{pmatrix}$, $0 \leqslant \beta < p^{\ell-1}$. From (3.4),

$$T = \left\{ \begin{pmatrix} a & p\beta + p^m c \\ b & a + p^m d \end{pmatrix} : a \in (\mathbb{Z}/p^\ell\mathbb{Z})^\times,\ 0 \leqslant b < p^\ell,\ 0 \leqslant c, d < p^m \right\} = K_m S$$

where $S = \{ \begin{pmatrix} a & p\beta y \\ y & a \end{pmatrix} \}$. Again S is abelian, and we can extend ϕ_A to a character ψ of T. We have $|T| = (p^\ell - p^{\ell-1}) p^\ell p^{2m} = (p-1)p^{3\ell-1}$ and the degree of $\chi = \mathrm{Ind}_T^G$ is $|G : T| = p^{\ell-2}(p^2-1)$.

3.3. The odd case

If $l = 2m+1$ is odd, the construction is somewhat more complicated. In this case K_m is not abelian; we start with the abelian normal subgroup K_{m+1} and define ϕ_A on K_{m+1} using the same formula (3.2) as before. We will have $T = \mathrm{Stab}_G(\phi_A) = K_m S$ for some subgroup S, depending the choices of A. Unlike the even case, we cannot extend ϕ_A to a linear character of T.

3.3.1. Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. From (3.4) the stabilizer T of ϕ_A is equal to

$$\left\{ \begin{pmatrix} a & p^m b \\ p^m c & d \end{pmatrix} : a, d \in (\mathbb{Z}/p^\ell\mathbb{Z})^\times,\ 0 \leqslant b, c < p^{m+1} \right\} = K_m S$$

where S is as in the even case, (3.5). Then $|T| = (p^\ell - p^{\ell-1})^2 p^{\ell+1} = p^{3\ell-1}(p-1)^2$. Denote

$$N = \left\{ \begin{pmatrix} 1 + p^m a & p^{m+1} b \\ p^m c & 1 + p^m d \end{pmatrix} \right\}, \qquad T_0 = \left\{ \begin{pmatrix} a & p^{m+1} b \\ p^m c & d \end{pmatrix} \right\}, \qquad T = \left\{ \begin{pmatrix} a & p^m b \\ p^m c & d \end{pmatrix} \right\}.$$

Then N is normal in T, and we can extend ϕ to a character ϕ' of N by

$$\phi' \begin{pmatrix} 1 + p^m a & p^{m+1} b \\ p^m c & 1 + p^m d \end{pmatrix} = \lambda'(1 + p^m a)$$

where λ' is a linear character of $(\mathbb{Z}/p^\ell\mathbb{Z})^\times$ such that $\lambda'(1 + p^m a) = \lambda(p^m a)$. We have $\mathrm{Stab}_T(\phi') = T_0$ and we can extend ϕ' to a character ψ_0 of T_0 by defining

$$\psi_0 \begin{pmatrix} a & p^{m+1} b \\ p^m c & d \end{pmatrix} = \lambda'(a).$$

Then let $\psi = \mathrm{Ind}_{T_0}^T \psi_0$ be the induced character, of degree p, of T. From Clifford theory, ψ is an irreducible character of T, and $(\psi|_{K_{m+1}}, \phi) = p$, so $\chi = \mathrm{Ind}_T^G \psi$ is an irreducible character of G; its degree is $p|G : T| = (p+1)p^{\ell-1}$, the same as in the even case.

Pictorially, have the following construction:

$$\begin{array}{ccccccccc} K_{m+1} & \longrightarrow & N & \longrightarrow & T_0 & \longrightarrow & T & \longrightarrow & G \\ \phi & \xrightarrow{\text{ext}} & \phi' & \xrightarrow{\text{ext}} & \psi_0 & \xrightarrow{\text{ind}} & \psi & \xrightarrow{\text{ind}} & \chi \end{array}.$$

3.3.2. Let $A = \begin{pmatrix} & \epsilon \\ 1 & \end{pmatrix}$. Define $\phi = \phi_A$ on K_{m+1}, as before, by $\phi_A(1 + p^{m+1}B) = \lambda(\mathrm{tr}(p^{m+1}AB))$, so

$$\phi_A \begin{pmatrix} 1 + p^{m+1}a & p^{m+1}b \\ p^{m+1}c & 1 + p^{m+1}d \end{pmatrix} = \lambda\big(p^{m+1}b + \epsilon p^{m+1}c\big). \tag{3.6}$$

From (3.4) the stabilizer of ϕ_A is

$$T = \mathrm{GL}\big(2, \mathbb{Z}/p^\ell\mathbb{Z}\big) \cap \left\{ \begin{pmatrix} x + p^m a & \epsilon y + p^m b \\ y + p^m c & x + p^m a \end{pmatrix} \right\} = K_m S, \quad S = \mathrm{GL}\big(2, \mathbb{Z}/p^\ell\mathbb{Z}\big) \cap \left\{ \begin{pmatrix} x & \epsilon y \\ y & x \end{pmatrix} \right\}.$$

Denote

$$N_i = K_i(K_1 \cap S)\left\{ \begin{pmatrix} a & \\ & a \end{pmatrix} \right\}.$$

The process to construct ψ is as follows. Let $H = N_{m+1}\big\langle \begin{pmatrix} 1+p^m & \\ & 1 \end{pmatrix} \big\rangle$. We extend ϕ_A to a character ϕ' of N_{m+1}, then to a character ϕ'' of H, then induce to N_m, then extend to T. Pictorially,

$$\begin{array}{ccccccccc} K_{m+1} & \longrightarrow & N_{m+1} & \longrightarrow & H & \longrightarrow & N_m & \longrightarrow & T \\ \phi_A & \xrightarrow{\text{ext}} & \phi' & \xrightarrow{\text{ext}} & \phi'' & \xrightarrow{\text{ind}} & \theta & \xrightarrow{\text{ext}} & \psi \end{array}.$$

Since N_m/N_{m+1} is abelian, any subgroup of N_m containing N_{m+1} is normal. Thus $H \lhd N_m$ and the index is p. Here are the details of this process:

(i) Extend ϕ to ϕ'.

Our first attempt is to define ϕ' on $K_1 \cap S$ using formula (3.6) which we used for ϕ_A, namely

$$\phi'\big(I + p(xI + yA)\big) = \lambda\big(\mathrm{tr}\big(p(xI + yA)A\big)\big) = \lambda(2py\epsilon).$$

However this does not preserve multiplication, since

$$\begin{aligned} &\phi'\big(I + p(x_1 I + y_1 A)I + p(x_2 I + y_2 A)\big) \\ &\quad = \phi'\big(I + p(x_1 + x_2)I + p(x_1 y_2 + x_2 y_1)A + p(y_1 + y_2)A^2\big) \\ &\quad = \lambda\big(2p(x_1 y_2 + x_2 y_1)\epsilon + 2p(y_1 + y_2)\epsilon\big). \end{aligned}$$

Note that multiplication would be preserved if either x_1 and x_2 are both divisible by p^{2m} or y_1 and y_2 are both divisible by p^{2m}. So we define ϕ' on the group $K_a = \big\{ \begin{pmatrix} 1+p^{2m}x & py\epsilon \\ py & 1+p^{2m}x \end{pmatrix} \big\}$ and $K_b = \big\{ \begin{pmatrix} 1+px & p^{2m}y\epsilon \\ p^{2m}y & 1+px \end{pmatrix} \big\}$ using (3.6) namely,

$$\phi' \begin{pmatrix} 1 + p^{2m}x & py\epsilon \\ py & 1 + p^{2m}x \end{pmatrix} = \lambda(2py\epsilon), \qquad \phi' \begin{pmatrix} 1 + px & p^{2m}y\epsilon \\ p^{2m}y & 1 + px \end{pmatrix} = \lambda\big(2p^{2m}y\epsilon\big).$$

Since $K_{m+1} \subset K_a \cap K_b$ then $|K_a K_b| = |K_1 \cap S|$ so $K_1 \cap S = K_a K_b$ and we can define the homomorphism ϕ' on $K_1 \cap S$ by

$$\phi'(gh) = \phi'(g)\phi'(h), \quad g \in K_a,\ h \in K_b.$$

Note that $\phi_A\left(\begin{smallmatrix}1+px & \\ & 1+px\end{smallmatrix}\right) = \lambda(0) = 1$; so we can define ϕ' to be trivial on all central elements $\left(\begin{smallmatrix}a & \\ & a\end{smallmatrix}\right)$ in N_{m+1}. It is clear that ϕ' is an extension of ϕ_A.

(ii) Extend ϕ' of N_{m+1} to ϕ'' of H.

We only need to define ϕ'' on $\left\langle\left(\begin{smallmatrix}1+p^m & \\ & 1\end{smallmatrix}\right)\right\rangle$. This can be done by first defining ϕ'' to be trivial on $\left(\begin{smallmatrix}1+p^m & \\ & 1\end{smallmatrix}\right)$.

(iii) Induction from ϕ'' to θ.

It is easy to find an element in N_m that does not stabilize ϕ'' and since the index of H in N_m is p, we have $\mathrm{Stab}_{N_m}(\phi'') = H$. Then Clifford's theorem tells us that $\theta = \mathrm{Ind}_H^{N_m} \phi''$ is irreducible.

(iv) Extend θ to ψ.

Since θ is stable under T and T/N_m is cyclic, then there exists an extension ψ.

3.3.3. Let $A = \left(\begin{smallmatrix}0 & p^j\beta \\ 1 & 0\end{smallmatrix}\right)$ where β is a unit of $\mathbb{Z}/p^\ell\mathbb{Z}$, and define ϕ_A on K_{m+1} again by $\phi(1+p^{m+1}B) = \lambda(\mathrm{tr}(p^{m+1}AB))$, so

$$\phi\begin{pmatrix}1+p^{m+1}a & p^{m+1}b \\ p^{m+1}c & 1+p^{m+1}d\end{pmatrix} = \lambda\big(p^{m+1}b + p^{m+j+1}c\beta\big). \tag{3.7}$$

$$T = \mathrm{Stab}_G(\phi_A) = \left\{\begin{pmatrix}a & p^j\beta b + p^m c \\ b & a + p^m d\end{pmatrix}\right\} = K_m S, \quad \text{where } S = \left\{\begin{pmatrix}w & p^j\beta y \\ y & w\end{pmatrix}\right\}.$$

Then $|T| = (p^\ell - p^{\ell-1})p^\ell p^{2(m+1)} = (p-1)p^{3\ell}$. Define $N = \left\{\left(\begin{smallmatrix}1+p^m a & p^{m+1}b \\ p^{m+1-j}c & 1+p^m d\end{smallmatrix}\right)\right\}$, so N is a normal subgroup of T, and we can extend ϕ_A to a character ϕ' of N similar to formula (3.7),

$$\phi'\begin{pmatrix}1+p^m a & p^{m+1}b \\ p^{m+1-j}c & 1+p^m d\end{pmatrix} = \lambda\big(p^{m+1}b + p^{m+1}c\beta\big).$$

The stabilizer of ϕ' is

$$T_0 = \left\{\begin{pmatrix}a & p^j\beta b + p^{m+1}c \\ b & a + p^m d\end{pmatrix}\right\} = NS.$$

Since S is abelian, we can extend ϕ' to a character ψ_0 of T_0 of degree 1. Let $\psi = \mathrm{Ind}_{T_0}^T \psi_0'$, so ψ is an irreducible character of T of degree p. Then $\chi = \mathrm{Ind}_T^G \psi$ is an irreducible character of G of degree $p|G:T| = (p^2-1)p^{\ell-2}$, as in the even case.

To summarize, we have the following table:

A	S	$\deg\chi$
$\left(\begin{smallmatrix}1 & 0 \\ 0 & 0\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}w & 0 \\ 0 & z\end{smallmatrix}\right)$	$(p+1)p^{\ell-1}$
$\left(\begin{smallmatrix}0 & \epsilon \\ 1 & 0\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}w & \epsilon y \\ y & w\end{smallmatrix}\right)$	$(p-1)p^{\ell-1}$
$\left(\begin{smallmatrix}0 & p\beta \\ 1 & 0\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}w & p\beta y \\ y & w\end{smallmatrix}\right)$	$(p^2-1)p^{\ell-2}$

3.4. The number of characters

In the last section, we constructed three kinds of irreducible characters of G. Now, we want to count the number of each kind and see that we actually have all the irreducible characters of each degree.

Theorem 3.1. *The non-linear irreducible characters of* $\mathrm{GL}(2, \mathbb{Z}/p^\ell\mathbb{Z})$ *which do not come from* $\mathrm{GL}(2, \mathbb{Z}/p^{\ell-1}\mathbb{Z})$ *are given as follows*:

deg χ	*number of* χ *of this degree*
$(p+1)p^{\ell-1}$	$\frac{1}{2}(p-1)^3p^{2\ell-3}$
$(p-1)p^{\ell-1}$	$\frac{1}{2}(p-1)(p^2-1)p^{2\ell-3}$
$(p^2-1)p^{\ell-2}$	$(p-1)p^{2\ell-2}$

Proof. For each of the matrices A used in the previous section, we will consider somewhat more general matrices A' to construct new characters $\phi_{A'}$. These will have the same degrees and stabilizers as ϕ_A. We will count the number of non-conjugate matrices we construct of type A', and count how many characters ψ_A of the stabilizer T which restrict to $\phi_{A'}$. This will give us the numbers of characters of each degree, as in the statement of the theorem. To show that these are all the characters, we will add up the sums of the squares of the degrees.

For the first part of the proof, assume that $\ell = 2m$ is even. In the first case where $A = \begin{pmatrix} 1 & \\ & \end{pmatrix}$, we consider more general matrices

$$A_{\alpha,k} = \alpha I + kA = \begin{pmatrix} \alpha + k & \\ & \alpha \end{pmatrix}, \quad \text{where } 0 \leqslant \alpha < p^m,\ 1 \leqslant k < p^m,\ p \nmid k.$$

We define $\phi_{A_{\alpha,k}}$ on K_m using the same formula (3.2). It is clear that $\mathrm{Stab}_G(\phi_{A_{\alpha,k}}) = \mathrm{Stab}_G(\phi_A) = T$ and we can also extend $\phi_{A_{\alpha,k}}$ to T. Therefore, we will get irreducible characters of G with degree $(p+1)p^{2m-1}$. Notice that $\begin{pmatrix} \alpha+k & \\ & \alpha \end{pmatrix}$ is conjugate to $\begin{pmatrix} \alpha & \\ & \alpha+k \end{pmatrix}$ hence the total number of non-conjugate $A_{\alpha,k}$, and so of non-conjugate $\phi_{A_{\alpha,k}}$ is

$$\frac{1}{2}p^m\left(p^m - p^{m-1}\right) = \frac{1}{2}p^{2m-1}(p-1) = \frac{1}{2}(p-1)p^{\ell-1}.$$

Multiplying by the number of extensions from K_m to T, which is $(p-1)^2p^{\ell-2}$ in this case, gives us $\frac{1}{2}(p-1)^3p^{2\ell-3}$ distinct irreducible characters of degree $(p+1)p^{\ell-1}$.

In the second case when $A = \begin{pmatrix} & \epsilon \\ 1 & \end{pmatrix}$, we replace A by

$$A_{\alpha,\epsilon} = \alpha I + A = \begin{pmatrix} \alpha & \epsilon \\ 1 & \alpha \end{pmatrix} \quad \text{where } 0 \leqslant \alpha < p^m,\ 0 < \epsilon < p^m \text{ and } \epsilon \text{ is a non-square unit.}$$

Notice that $A_{\alpha,\epsilon}$ is conjugate to $A_{\alpha',\epsilon'}$ if and only if $\alpha = \alpha', \epsilon = \epsilon'$. We get $\frac{1}{2}(p^m - p^{m-1})p^m = \frac{1}{2}(p-1)p^{\ell-1}$ non-conjugate characters of the form $\phi_{A_{\alpha,\epsilon}}$. Each of these has $|T : K_m|$ extensions to T; $|T : K_m| = p^{3\ell-2}(p^2-1)/p^{2\ell} = p^{\ell-2}(p^2-1)$. We get $\frac{1}{2}(p^2-1)(p-1)p^{2\ell-3}$ distinct irreducible characters of G of degree $(p-1)p^{\ell-2}$.

In the third case when $A = \begin{pmatrix} & p\beta \\ 1 & \end{pmatrix}$, we use

$$A_{\alpha,\beta} = \alpha I + A = \begin{pmatrix} \alpha & p\beta \\ 1 & \alpha \end{pmatrix} \quad \text{where } 0 \leqslant \alpha < p^m,\ 0 \leqslant \beta < p^{m-1}.$$

By counting the number of α and β, we have $p^m p^{m-1} = p^{\ell-1}$ non-conjugate $\phi_{\alpha,\beta}$ of K_m. The number of extensions of each to T is $|T : K_m| = p^{3\ell-1}(p^2-1)/p^{2\ell} = p^{\ell-1}(p^2-1)$, giving us $p^{2\ell-2}(p-1)$ distinct irreducible characters of G of degree $(p^2-1)p^{\ell-2}$.

Now assume that $\ell = 2m+1$ is odd.

First, let $A = \begin{pmatrix} 1 & \\ & \end{pmatrix}$. Replace A by more general matrices

$$A_{\alpha,k} = \alpha I + kA = \begin{pmatrix} \alpha + k & \\ & \alpha \end{pmatrix}, \quad \text{where } 0 \leqslant \alpha < p^{m+1},\ 0 < k < p^{m+1},\ p \nmid k.$$

Each $\phi_{A_{\alpha,k}}$ can be extended to $\phi'_{A_{\alpha,k}}$ on N, giving $\frac{1}{2}(p^{m+1} - p^m)p^{m+1} = \frac{1}{2}p^\ell(p-1)$ non-conjugate linear characters $\phi_{A_{\alpha,k}}$ of N. The number of extensions from N to T_0 is $|T_0|/|N| = p^{\ell-3}(p-1)^2$, and we get $\frac{1}{2}p^\ell(p-1)p^{\ell-3}(p-1)^2 = \frac{1}{2}(p-1)^3p^{2\ell-3}$ irreducible characters of G with degree $(p+1)p^{\ell-1}$.

Second, consider the case $A = \begin{pmatrix} & \epsilon \\ 1 & \end{pmatrix}$. Replace A by

$$A_{\alpha,\epsilon} = \alpha I + A = \begin{pmatrix} \alpha & \epsilon \\ 1 & \alpha \end{pmatrix} \quad \text{where } 0 \leqslant \alpha < p^m,\ 0 < \epsilon < p^m \text{ and } \epsilon \text{ is a non-square unit,}$$

we have $\frac{1}{2}(p^m - p^{m-1})p^m = \frac{1}{2}(p-1)p^{\ell-2}$ non-conjugate $\phi_{A_{\alpha,\epsilon}}$ on K_{m+1}. The number of extensions from $\phi_{A_{\alpha,\epsilon}}$ to ϕ' is $|N_{m+1}|/|K_{m+1}| = (p-1)p^{\ell-1}$, the number of extensions from θ to ψ is $|T|/|N_m| = p+1$. Therefore, we get $\frac{1}{2}(p-1)p^{\ell-2}(p-1)p^{\ell-1}(p+1) = \frac{1}{2}(p-1)(p^2-1)p^{2\ell-3}$ distinct irreducible characters of G with degree $(p^2-1)p^{\ell-2}$.

Now let us consider the third case when $A = \begin{pmatrix} & p\beta \\ 1 & \end{pmatrix}$. We take more general matrices

$$A_{\alpha,\beta} = \alpha I + A = \begin{pmatrix} \alpha & p\beta \\ 1 & \alpha \end{pmatrix} \quad \text{where } 0 \leqslant \alpha < p^{m+1},\ 0 \leqslant \beta \leqslant p^m,$$

to define $\phi'_{A_{\alpha,\beta}}$ on K_{m+1}. The number of such $\phi_{\alpha,\beta}$ is $p^{m+1}p^m = p^\ell$ and the number of extensions in this case is $|T_0|/|N| = (p-1)p^{\ell-2}$. This gives $(p-1)p^{2\ell-2}$ irreducible characters of G with degree $(p-1)p^{\ell-1}$.

Let S_1 denote the sum of the squares of the character degrees we have constructed. We have

$$S_1 = p^{4\ell-6}\big(p^2-1\big)(p-1)\big(p^3-1\big).$$

We also have to find the sum of the squares of the characters coming from $\mathrm{GL}(2, \mathbb{Z}/p^{\ell-1}\mathbb{Z}) = G_{\ell-1}$. Let Y denote the set of irreducible characters of $G = G_\ell$ inflated from $G_{\ell-1}$, that is, having $K_{\ell-1}$ contained in their kernel. Let L_ℓ be the set of linear characters of G_ℓ, and let X be the set of characters of G_ℓ of the form $\mu\chi$ where $\mu \in L_\ell$ and $\chi \in Y$. If $\mu\chi = \mu'\chi'$, where $\mu, \mu' \in L_\ell$ and $\chi, \chi' \in Y$, then $\chi = \mu^{-1}\mu'\chi'$, so $\mu^{-1}\mu'$ has kernel containing $K_{\ell-1}$. It follows that if $\chi \in Y$, then the number of distinct characters of the form $\mu\chi \in X$ is equal to $|L_\ell|/|L_{\ell-1}|$. The derived group of G_ℓ is $SL(2, \mathbb{Z}/p^\ell\mathbb{Z})$, and L_ℓ is isomorphic to $G_\ell/SL(2, \mathbb{Z}/p^\ell\mathbb{Z})$ which is isomorphic to $(\mathbb{Z}/p^\ell\mathbb{Z})^\times$. Since $|(\mathbb{Z}/p^\ell\mathbb{Z})^\times|/|(\mathbb{Z}/p^{\ell-1}\mathbb{Z})^\times| = p$, we see that there are p distinct characters $\mu\chi$ in X, for each $\chi \in Y$. Then the sum of the squares of the degrees of the characters in X is

$$S_2 = p|G_{\ell-1}| = p^{4\ell-6}\big(p^2-1\big)(p-1).$$

Then

$$S_1 + S_2 = p^{4\ell-6}\big(p^2-1\big)(p-1)\big(p^3-1+1\big) = p^{4\ell-3}\big(p^2-1\big)(p-1)$$

which is the order of G_ℓ. It follows that we have accounted for all the characters of G_ℓ. $\square$

3.5. About character values

In the proof of the last theorem, we replaced the matrix A by related matrices of the form A'. We want to see how the character values are affected by this change.

The matrices $A' = \alpha A$ give linear characters $\phi_{A'}$ of K_j, where j is m or $m+1$, depending on whether $\ell = 2m$ or $2m+1$. In either case, $2j \geqslant \ell$. Let $\lambda^\alpha(g) = \lambda(\alpha g)$. If $A' = \alpha I + A$, then

$$\begin{aligned}\phi_{A'}\big(I + p^j B\big) &= \lambda\big(\mathrm{tr}\big(p^j(\alpha I + A)B\big)\big)\\ &= \lambda\big(\mathrm{tr}\big(p^j \alpha B + p^j AB\big)\big)\\ &= \lambda^\alpha\big(\mathrm{tr}\big(p^j B\big)\big)\lambda\big(p^j AB\big)\\ &= \lambda^\alpha\big(\mathrm{tr}\big(p^j B\big)\big)\phi_A\big(I + p^j B\big).\end{aligned}$$

Let $B = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$; since $2j \geqslant \ell$, then $\det(I + p^j B) = 1 + p^j(a+d) = 1 + p^j \operatorname{tr}(B)$. Let $\mu : (\mathbb{Z}/p^\ell\mathbb{Z})^\times \to \mathbb{C}^\times$ be a character of $(\mathbb{Z}/p^\ell\mathbb{Z})^\times$ such that

$$\lambda^\alpha\big(\mathrm{tr}\big(p^j B\big)\big) = \mu\big(\det\big(I + p^j B\big)\big).$$

Therefore,

$$\phi_{A'} = (\mu \circ \det)\phi_A.$$

Notice that $\mu \circ \det$ is a linear character of G, and hence it is stable under G; we have $\mathrm{Stab}_G(\phi_{A'}) = \mathrm{Stab}_G(\phi_A)$ and $(\mu \circ \det)\psi$ is an extension of $\phi_{A'}$ provided that ψ is an extension of ϕ_A. It is clear that

$$\chi_{A'} = \mathrm{Ind}_T^G(\mu \circ \det)\psi = (\mu \circ \det)\,\mathrm{Ind}_T^G \psi = (\mu \circ \det)\chi.$$

From this formula, it follows that we can deduce the character values of $\chi_{A'}$ from those of χ_A.

We have also used $A' = \alpha A + kI$, where k is a unit of $\mathbb{Z}/p^\ell\mathbb{Z}$. In this case

$$\phi_{A'} = (\mu \circ \det)\phi_A^k,$$

and again the character values of $\chi_{A'}$ can be gotten from those of χ_A.

4. Values for characters with degree $(p+1)p^{\ell-1}$

In this section, we will first find values for characters of degree $(p+1)p^{\ell-1}$ by parabolic induction. Then using Clifford theory we construct irreducible characters with the same degree and show that these two kinds of irreducible characters are the same.

4.1. Character values by parabolic induction

Here ℓ can by any positive integer, and let $R = \mathbb{Z}/p^\ell\mathbb{Z}$. Let $B = \left\{\left(\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}\right)\right\}$ be the Borel subgroup of $\mathrm{GL}(R)$. Let λ' be an injective character of $R^\times$, and let ϕ be the character of B given by $\phi\left(\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}\right) = \lambda'(a)$, and let $\chi' = \mathrm{Ind}_B^G \phi$.

Lemma 4.1. $\mathrm{Ind}_B^G \phi'$ *is irreducible.*

Proof. It suffices to show $(\chi', \chi') = (\phi, \chi'|_B) = 1$. By Mackey's theorem [1, p. 74] we have

$$\chi'|_B = \bigoplus_{G=\bigcup BgB} \mathrm{Ind}^B_{gBg^{-1}\cap B}\big(\phi^g\big), \quad \text{where } \phi^g\big(gXg^{-1}\big) = \phi(X),\ X \in B.$$

In order to calculate $(\phi, \chi'|_B)$, we want to look at $(\phi, \mathrm{Ind}^B_{gBg^{-1}\cap B}(\phi^g)) = (\phi|_{gBg^{-1}\cap B}, \phi^g)$ for each double coset representative g of B. Pick the double coset representatives of B to be

$$g = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad g_i = \begin{pmatrix} 1 & 0 \\ p^i & 1 \end{pmatrix}, \quad 1 \leqslant i \leqslant \ell.$$

Let $X = \left(\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}\right)$, we have $gXg^{-1} = \left(\begin{smallmatrix} c & 0 \\ b & a \end{smallmatrix}\right)$. Therefore,

$$gBg^{-1} \cap B = \left\{\begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix}\right\}, \quad \phi\begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix} = \lambda(a), \quad \phi^g\begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix} = \lambda(c).$$

Since $\phi \neq \phi^g$ and they both have degree 1, we know $(\phi|_{gBg^{-1}\cap B}, \phi^g) = 0$. Similarly we have

$$g_iBg_i^{-1} \cap B = \left\{\begin{pmatrix} p^{\ell-i}a + c & b \\ 0 & p^ib + c \end{pmatrix}\right\}, \quad \text{and} \quad \big(\phi|_{g_iBg_i^{-1}\cap B}, \phi^{g_i}\big) = 0, \quad \text{when } 1 \leqslant i < \ell.$$

It is clear that $(\phi|_{g_\ell Bg_\ell^{-1}\cap B}, \phi^{g_\ell}) = 1$ and hence, $(\phi, \chi'_B) = (\chi', \chi') = 1$. $\square$

To find the character value on an arbitrary conjugacy Class C, we use the following formula:

$$\chi'(C) = \frac{[G:B]}{|C|} \sum_{a\in R^\times} \lambda(a) \left|\left\{(b,c) \in R^2\colon \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \in B \cap C\right\}\right|. \tag{4.1}$$

Evaluating this for all C requires us to analyze how B breaks up into conjugacy class of G. Let $A = \left(\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}\right)$ be an arbitrary element of B, and define nonnegative integers $i \leqslant \ell$ and $j \leqslant \ell$ by $a - c \in p^iR^\times$ and $b \in p^jR^\times$. If $i = j = \ell$ then $a = c$ and $b = 0$, so that $A \in I_\alpha$. In view of the section on conjugacy classes of $\mathrm{GL}(2, R)$, we can identify the conjugacy class of A by its trace and the determinant of a related matrix in $\mathrm{GL}(2, R)$. Thus,

$$\begin{aligned} &A \in D_{iac} \quad \text{if } i \leqslant j \text{ and } i < \ell \\ &\quad\Rightarrow \quad \chi'(D_{iac}) = \frac{(p+1)p^{\ell-1}}{(p+1)p^{2\ell-2i-1}} p^{\ell-i}\big(\lambda'(a) + \lambda'(c)\big) = p^i\big(\lambda'(a) + \lambda'(c)\big). \end{aligned}$$

Moreover

$$A \in B_{j(\frac{a+c}{2})(\frac{(a-c)^2}{4p^{2j+1}})} \quad \text{if } i > j.$$

Since the conjugacy class type $C_{i\alpha\beta}$ does not occur, the character values on this class are 0. Suppose that $j < i$ and $j \leqslant \ell - 2$. Then

$$\left\{\begin{pmatrix} a + dp^{\ell-1} & b \\ 0 & c - dp^{\ell-1} \end{pmatrix} : d \in R/pR\right\}$$

is a set of conjugate matrices. Their total contribution to the sum (4.1) is

$$\sum_{d=0}^{p-1} \lambda'\bigl(a + dp^{\ell-1}\bigr) = 0.$$

Thus $\chi'(B_{j\alpha\beta}) = 0$ when $j \leqslant \ell - 2$. The only remaining class is $B_{(\ell-1)\alpha 0}$, for which (4.1) reads

$$\chi'(B_{(\ell-1)\alpha 0}) = \frac{\deg \chi'}{(p+1)(p-1)} \sum_{a \in R^\times} \lambda'(a) \bigl|\bigl\{(b,c) \in R^2\colon a = c = \alpha,\ b \in p^{\ell-1}R^\times\bigr\}\bigr| = p^{\ell-1}\lambda'(a).$$

4.2. Character values by Clifford's theorem

Now we will use Clifford's theorem to construct irreducible characters of $\mathrm{GL}(2, \mathbb{Z}/p^\ell\mathbb{Z})$ with degree $p^{\ell-1}(p+1)$ and we will see that they have the same character values as χ' in the last section. We first assume $\ell = 2m$ and will discuss the odd case later.

Let $\lambda' : (\mathbb{Z}/p^{2m}\mathbb{Z})^\times \to \mathbb{C}^\times$ be injective. From Section 3.2.1, we have the character $\chi = \mathrm{Ind}_T^G \psi$ where

$$\psi \begin{pmatrix} a & p^m b \\ p^m c & d \end{pmatrix} = \lambda'(a).$$

Next, we want to find the character values of χ.

Lemma 4.2. *Suppose $u, v \in (\mathbb{Z}/p^{2m}\mathbb{Z})^\times$ and $m \leqslant k < l$, then $\sum_{0 \leqslant t < p^m} \lambda'(u + p^k t v) = 0$.*

Proof.

$$\begin{aligned} \sum_{0 \leqslant t < p^m} \lambda'\bigl(u + p^k t v\bigr) &= \sum_{0 \leqslant t < p^m} \lambda'(u)\lambda'\bigl(1 + p^k t v u^{-1}\bigr) \\ &= \lambda'(u) \sum_{0 \leqslant t < p^m} \lambda'\bigl(1 + p^k t\bigr) \\ &= \lambda'(u) \sum_{0 \leqslant t < p^m} \lambda'\bigl(1 + p^k\bigr)^t \\ &= 0. \qquad \square \end{aligned}$$

Pick the coset representatives of T to be

$$E_{xy} = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix}, \qquad F_{xz} = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} pz & 1 \\ 1 & pz \end{pmatrix}, \quad 0 \leqslant x, y,\ pz < p^m.$$

We first evaluate $\psi(C_{i\alpha\beta})$. Since $T \cap C_{i\alpha\beta} = \varnothing$ if $i < m$, we assume $l > i \geqslant m$. Notice that

$$E_{xy} \begin{pmatrix} \alpha & p^i\epsilon\beta \\ p^i\beta & \alpha \end{pmatrix} E_{xy}^{-1} = \begin{pmatrix} \alpha - p^i\epsilon\beta y + p^i\beta x(1 - \epsilon y^2) & p^m * \\ p^m * & * \end{pmatrix},$$

$$F_{xz} \begin{pmatrix} \alpha & p^i\epsilon\beta \\ p^i\beta & \alpha \end{pmatrix} F_{xz}^{-1} = \begin{pmatrix} [\alpha + p^{i+1}\beta z(\epsilon - 1)(1 - p^2z^2)^{-1}] + p^i x\beta(\epsilon - p^2z^2)(1 - p^2z^2)^{-1} & * \\ p^m * & * \end{pmatrix},$$

we have

$$\chi(C_{i\alpha\beta}) = \sum_{0\leqslant x,y<p^m} \lambda'\left[\left(\alpha - p^i\epsilon\beta y\right) + p^i x\beta\left(1-\epsilon y^2\right)\right]$$
$$+ \sum_{0\leqslant pz,x<p^m} \lambda'\left\{\left[\alpha + p^{i+1}\beta z(\epsilon-1)\left(1-p^2z^2\right)^{-1}\right] + p^i x\beta\left(\epsilon - p^2z^2\right)\left(1-p^2z^2\right)^{-1}\right\}.$$

By Lemma 4.2,

$$\sum_{0\leqslant x<p^m} \lambda'\left[\left(\alpha - p^i\epsilon\beta y\right) + p^i x\beta\left(1-\epsilon y^2\right)\right] = 0,$$
$$\sum_{0\leqslant x<p^m} \lambda'\left\{\left[\alpha + p^{i+1}\beta z(\epsilon-1)\left(1-p^2z^2\right)^{-1}\right] + p^i x\beta\left(\epsilon - p^2z^2\right)\left(1-p^2z^2\right)^{-1}\right\} = 0.$$

Therefore,

$$\chi(C_{i\alpha\beta}) = 0.$$

Using a similar method and applying Lemma 4.2 throughout the calculation, we have

$$\chi(B_{(\ell-1)\alpha\beta}) = p^{\ell-1}\lambda'(a) \quad \text{and} \quad \chi(D_{i\alpha\beta}) = p^i\left[\lambda'(\alpha) + \lambda'(\delta)\right].$$

Compare the values of χ we constructed here with χ' by parabolic induction in the last subsection; we observe that χ and χ' take the same nonzero values. Since $\deg(\chi) = \deg(\chi')$ and χ is irreducible, we have

$$\left(\chi, \chi'\right) = \left(\chi', \chi'\right) = 1.$$

Therefore, $\chi = \chi'$, and this is another way to show that χ' in the last section is irreducible.

Now we consider the odd case. From Section 3.3.1, we have the irreducible character $\chi = \mathrm{Ind}_T^G(\psi) = \mathrm{Ind}_{T_0}^G(\psi_0)$. We find character values of $\mathrm{Ind}_{T_0}^G(\psi_0)$ just as in the even case, and we get the same character values as we got using parabolic induction.

5. Values for $(p-1)p^{\ell-1}$-degree characters

In this section, we will construct the irreducible characters of G with degree $(p-1)p^{\ell-1}$ and find the character values. We first find character values on $K_{\ell-i}$, $1 \leqslant i \leqslant \frac{\ell}{2}$, and then work on the remaining character values in two cases depending on whether ℓ is even or odd.

5.1. Character values of elements in $K_{\ell-i}$, $1 \leqslant i \leqslant \frac{\ell}{2}$

Let S denote the subgroup of $\mathrm{GL}(2, \mathbb{Z}/p^\ell\mathbb{Z})$ consisting of matrices of the form $\left(\begin{smallmatrix} a & b\epsilon \\ b & a \end{smallmatrix}\right)$.

Lemma 5.1. *The group S is the semi-direct product $S = (K_1 \cap S)\langle s_0\rangle$ where s_0 has order p^2-1; s_0^{p+1} has the form $\left(\begin{smallmatrix} c & 0 \\ 0 & c \end{smallmatrix}\right)$.*

Proof. From Lemma 3.2, $S/S\cap K_1 \cong S_1$, where S_1 is cyclic of order p^2-1. Let s_1 be an element of S whose image in S_1 generates S_1. Then $s_1^{p^2-1}$ belongs to $K_1 \cap S$, which is a p-group. So a suitable p-power s_0 of s_1 has order p^2-1, and the image of s_0 in S_1 generates S_1. Thus S is the semi-direct product $(K_1 \cap S)\langle s_0\rangle$. The subgroup S_0 of matrices in $\mathrm{GL}(2, \mathbb{Z}/p\mathbb{Z})$ of the form $\left(\begin{smallmatrix} c & 0 \\ 0 & c \end{smallmatrix}\right)$ is isomorphic to the units of $GF(p)$, and has order $p-1$. Modulo K_1, s_0^{p+1} belongs to S_0, so s_0^{p+1} has the form $\left(\begin{smallmatrix} c & 0 \\ 0 & c \end{smallmatrix}\right)y$ for some $y \in K_1 \cap S$. Since y has order a power of p and s_0^{p+1} has order prime to p, then $y = 1$, and the result is proved. □

Let A be the matrix $\left(\begin{smallmatrix} 0 & \epsilon \\ 1 & 0 \end{smallmatrix}\right)$ over the ring $R_i = \mathbb{Z}/p^i\mathbb{Z}$, $\lambda : R_i \to \mathbb{C}^\times$, and let ϕ_A be the corresponding character of $K_{\ell-i}$:

$$\phi_A \begin{pmatrix} 1 + p^{\ell-i}a & p^{\ell-i}b \\ p^{\ell-i} & 1 + p^{\ell-i}a \end{pmatrix} = \lambda\, \mathrm{tr}\left(\begin{pmatrix} 0 & \epsilon \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ 1 & a \end{pmatrix} \right) = \lambda(b + \epsilon).$$

Let S_i denote the subgroup of $\mathrm{GL}(2, R_i)$ consisting of matrices of the form $\left(\begin{smallmatrix} a & b\epsilon \\ b & a \end{smallmatrix}\right)$, so $|S_i| = (p^2-1)p^{2i-2}$.

Lemma 5.2. *The following set of cardinality $(p-1)p^{2i-1}$ includes exactly one representative from each right coset of S_i in $\mathrm{GL}(2, R_i)$:*

$$\left\{ \begin{pmatrix} 1 & c \\ & d \end{pmatrix} : c \in R_i,\ d \in R_i^\times \right\}.$$

Proof. It is easy to check that the above set actually forms a subgroup of $\mathrm{GL}(2, R_i)$ and the only element that lies in S_i is the identity. The number of elements in this subgroup is $(p^i - p^{i-1})p^i = p^{2i-1}$. On the other hand,

$$\left|\mathrm{GL}\big(2, \mathbb{Z}/p^i\mathbb{Z}\big)\right| = |K_1|\left|\mathrm{GL}(2, \mathbb{Z}/p\mathbb{Z})\right| = p^{4(i-1)}\big(p^2-1\big)\big(p^2-p\big),$$

so the index $|\mathrm{GL}(2, \mathbb{Z}/p^i\mathbb{Z}) : S_i| = p^{2i-1}$. So we have a complete list of coset representatives. □

Lemma 5.3. . *Let $z \in R_i^\times$. The number of solutions $(x, y) \in R_i^2$ of the equation $x^2 - \epsilon y^2 = z$ is $(p+1)p^{i-1}$.*

Proof. We claim that the map $\det : S_i \to R_i^\times$ is surjective. This is easily seen if $i = 1$. In general, the claim follows using the commutative diagram

$$\begin{array}{ccc} S_i & \longrightarrow & S_1 \\ \downarrow & & \downarrow \\ (\mathbb{Z}/p^i\mathbb{Z})^\times & \longrightarrow & (\mathbb{Z}/p\mathbb{Z})^\times \end{array}$$

where the horizontal maps are "mod p" and the vertical maps are det. The number of solutions to $x^2 - \epsilon y^2 = z$ is the number of matrices $\left(\begin{smallmatrix} x & \epsilon y \\ y & x \end{smallmatrix}\right)$ in S_i whose determinant is z. This number is $|S_i|/|R_i^\times| = (p+1)p^{i-1}$. □

Lemma 5.4. *If $\lambda : R_i^+ \to \mathbb{C}^\times$ is injective and $0 \leqslant j \leqslant i$ then*

$$\sum \left\{ \lambda(y) \colon y \in p^j R_i^\times \right\} = \begin{cases} 0, & \text{if } j < i-1, \\ -1, & \text{if } j = i-1, \\ 1, & \text{if } j = i. \end{cases}$$

The proof uses the fact that for any $y_0 \in R_i$ and $j < i$,

$$\sum\bigl\{\lambda(y)\colon\ y \equiv y_0 \pmod{p^j}\bigr\} = 0.$$

Suppose χ is any irreducible character of G whose restriction to $K_{\ell-i}$ contains copies of ϕ_A. For any $X \in K_{\ell-i}$, by Clifford's theorem,

$$\chi(X) = e\sum_{i=1}^{t}\phi_i(X),$$

where $\phi_1, \phi_2, \dots, \phi_t$ are the distinct conjugates of ϕ_A in G.

Choose

$$E_{cd} = \begin{pmatrix} 1 & c \\ & d \end{pmatrix}\colon\ c \in R_i,\ d \in R_i^{\times},$$

from Lemma 5.2, we have each $\phi_i = \phi_{E_{cd}^{-1}AE_{cd}}$ for some c, d, which implies $t = (p-1)p^{2i-1}$. Notice that each ϕ_i has degree 1, we have $e = \frac{\deg(\chi)}{(p-1)p^{2i-1}}$. Therefore,

$$\chi(X) = \frac{\deg(\chi)}{(p-1)p^{2i-1}}\sum_{c\in R_i}\sum_{d\in R_i^{\times}}\lambda\biggl(\operatorname{tr}\biggl(A\frac{E_{cd}XE_{cd}^{-1}}{p^{\ell-i}}\biggr)\biggr).$$

Now when $X = 1 + p^{\ell-i}\left(\begin{smallmatrix} a & b \\ 1 & a \end{smallmatrix}\right)$,

$$\begin{aligned} E_{cd}XE_{cd}^{-1} &= 1 + p^{\ell-i}\begin{pmatrix} 1 & \\ & d \end{pmatrix}\begin{pmatrix} 1 & c \\ & 1 \end{pmatrix}\begin{pmatrix} a & b \\ 1 & a \end{pmatrix}\begin{pmatrix} 1 & -c \\ & 1 \end{pmatrix}\begin{pmatrix} 1 & \\ & d^{-1} \end{pmatrix} \\ &= 1 + p^{\ell-i}\begin{pmatrix} a+c & d^{-1}(b-c^2) \\ d & a-c \end{pmatrix}. \end{aligned}$$

Therefore $\operatorname{tr}\bigl(A\frac{E_{cd}XE_{cd}^{-1}}{p^{\ell-i}}\bigr) = \epsilon d + d^{-1}(b-c^2)$, so that

$$\chi(X) = p^{\ell-2i}\sum_{c\in R_i}\sum_{d\in R_i^{\times}}\lambda\bigl(\epsilon d + d^{-1}\bigl(b-c^2\bigr)\bigr).$$

Define

$$P = \sum_{c\in R_i}\sum_{d\in R_i^{\times}}\lambda\bigl(\epsilon d + d^{-1}\bigl(b-c^2\bigr)\bigr)$$

so that $\chi(X) = p^{\ell-2i}P$. To evaluate this sum, we ask how many times $\epsilon d + d^{-1}(b-c^2)$ attains a particular value $x \in R_i$; that is, we want to know the number of solutions $(c, d) \in R_i \times R_i^{\times}$ to the equation

$$\begin{aligned} &\epsilon d + d^{-1}\bigl(b-c^2\bigr) = x \\ \Leftrightarrow\quad &\epsilon d^2 + b - c^2 = dx. \end{aligned} \tag{5.1}$$

We first identify solutions to (5.1) for which $d \in pR_i$. Later we will count solutions with $d \in R_i$ and could subtract to find P. Now if $d \in pR_i$ then x satisfies (5.1) if and only if $x + p^{i-1}$ does. Therefore the sum of $\lambda(x)$ over all solutions to (5.1) with $d \in pR_i$ is 0. Henceforth, we may let $d \in R_i$ and we will obtain the same sum P as if we restricted d to be a unit.

Manipulate (5.1) some more:

$$\begin{aligned} (5.1) \quad &\Leftrightarrow \quad \epsilon^2 d^2 + \epsilon b - \epsilon c^2 = \epsilon d x \\ &\Leftrightarrow \quad \left(\epsilon d - \frac{x}{2}\right)^2 - \left(\frac{x}{2}\right)^2 + \epsilon b - \epsilon c^2 = 0 \\ &\Leftrightarrow \quad d'^2 - \epsilon c^2 = y, \end{aligned} \tag{5.2}$$

where

$$d' = \epsilon d - \frac{x}{2} \quad \text{and} \quad y = \left(\frac{x}{2}\right)^2 - \epsilon b.$$

In counting solutions $(c, d') \in R_i^2$, it now becomes helpful that ϵ is not a square in either R_i or F_p. If $y = 0$ then we must have $d'^2 = 0 = c^2$, so that $c, d' \in p^{\lfloor \frac{i}{2} \rfloor} R_i$, and there are $p^{2(\ell - \lfloor \frac{i}{2} \rfloor)}$ solutions. If $y = p^{2k+1}u$ for some $u \in R_i^\times$ and some k, $0 < 2k+1 < i$, then there are no solutions because

$$\begin{aligned} d'^2 - \epsilon c^2 = 0 \quad &\Rightarrow \quad d'^2 \equiv 0 \equiv c^2 \pmod{p^{2k+1}} \\ &\Rightarrow \quad d', c \in p^{k+1} R_i \\ &\Rightarrow \quad y = d'^2 - \epsilon c^2 = 0 \pmod{p^{2k+2}} \end{aligned}$$

a contradiction.

Lastly, let $y = p^{2k}u$, $u \in (\mathbb{Z}/p^{i-2k}\mathbb{Z})^\times$, $0 \leqslant 2k < i$. By the same reasoning we must have $c, d' \in p^k R_i$. Let $p^k \widetilde{c} = c \bmod p^{i-2k}$ and $p^k \widetilde{d'} = d \bmod p^{i-2k}$, where $\widetilde{c}, \widetilde{d'} \in \mathbb{Z}/p^{i-2k}\mathbb{Z}$. Then (5.2) holds if and only if $\widetilde{d'}^2 - \epsilon \widetilde{c}^2 = u$. By Lemma 5.3, there are $(p+1)p^{i-2k-1}$ solutions $(\widetilde{c}, \widetilde{d'})$, and hence $(p+1)p^{i-1}$ solutions (c, d').

To summarize, let p^j be the highest power of p that divides y, if $y \neq 0$, and let $\rho(y)$ be the number of solutions $(c, d') \in R_i^2$ of (5.2). We have shown that $\rho(y) = p^{2(\ell - \lfloor \frac{i}{2} \rfloor)}$ if $y = 0$; $\rho(y) = (p+1)p^{i-1}$ if j is even, and $\rho(y) = 0$ if j is odd. From this, we can see that $\rho(uy) = \rho(y)$ for all $u \in R_i^\times$.

We still need to evaluate P:

$$P = \sum_{x \in R_i} \rho\left(\frac{x}{2}\right)^2 - \epsilon b \lambda(x) = \sum_{x \in R_i} \rho\bigl(x^2 - 4\epsilon b\bigr)\lambda(x).$$

If $b \in R_i^\times$ is a square, then ϵb is a non-square so that $((\frac{x}{2})^2 - \epsilon b) \in R_i^\times$ for all x; thus ρ always takes the same value $(p+1)p^{i-1}$, and $P = 0$.

If $b \in pR_i$ and $i \geqslant 2$ then we can ignore terms in P with $x \in pR_i$, because the terms for $x + yp^{i-1}$, $0 \leqslant y < p$, sum to 0. But the terms in P with $x \in R_i^\times$ also sum to 0 by Lemma 5.4. However, if $i = 1$ and $b \in pR_i = 0$ then by Lemmas 5.3 and 5.4, $P = \rho(0) - \rho(1) = -p$.

If $b \in R_i^\times$ is a non-square, then we have $\epsilon b = u^2$ for some $u \in R_i^\times$, so

$$P = \sum_{x \in R_i} \rho\bigl((x - 2u)(x + 2u)\bigr)\lambda(x).$$

We only need to consider the terms in the above summation with $x = p^j t \pm 2u$, $i-1 \leqslant j \leqslant i$, $t \in R_i^\times$, because the remaining terms will sum to 0. From the results of ρ we got above, we have

$$x = p^{i-1} t \pm 2u \quad \Rightarrow \quad \rho\big((x-2u)(x+2u)\big) = \rho\big(p^{i-1}\big)$$

and

$$x = \pm 2u \quad \Rightarrow \quad \rho\big((x-2u)(x+2u)\big) = \rho(0).$$

From Lemma 5.4, we have

$$\sum_{t \in R_i^\times} \lambda\big(p^{i-1} t \pm 2u\big) = \lambda(\pm 2u).$$

Therefore

$$P = \big(\rho(0) - \rho\big(p^{i-1}\big)\big)\big(\lambda(2u) + \lambda(-2u)\big)$$

in this case. And it is easy to check that $\rho(p^{i-1}) - \rho(0) = (-p)^i$.

We find that

$$\begin{aligned} &P = (-p)^i\big(\lambda(2u) + \lambda(-2u)\big) && \text{if } u^2 = \epsilon b \in R_i^\times, \\ &P = -p && \text{if } i = 1 \text{ and } b = 0, \\ &P = 0 && \text{otherwise}. \end{aligned}$$

Therefore, we have

$$\chi(I_\alpha) = \deg(\chi) = (p-1)p^{\ell-1},$$

$$\begin{aligned} \chi(C_{i\alpha\beta}) &= (-1)^i p^{\ell-i}\big(\lambda(2\epsilon\beta) + \lambda(-2\epsilon\beta)\big) \\ &= (-1)^i p^{\ell-i}\left(\phi_A\begin{pmatrix} \alpha & p^i\epsilon\beta \\ p^i\beta & \alpha \end{pmatrix} + \phi_A\begin{pmatrix} \alpha & -p^i\epsilon\beta \\ -p^i\beta & \alpha \end{pmatrix}\right), \end{aligned}$$

$$\begin{aligned} &\chi(B_{i\alpha\beta}) = (-p)^{\ell-1} \quad \text{if } i = \ell-1 \text{ and } 0 \text{ otherwise}, \\ &\chi(D_{i\alpha\delta}) = 0 \end{aligned}$$

valid when $i \geqslant \frac{\ell}{2}$.

5.2. Remaining values when $\ell = 2m$ is even

Lemma 5.5. *If i and j are positive integers and $\lambda' : (\mathbb{Z}/p^i\mathbb{Z})^+ \to \mathbb{C}^\times$ is an injective homomorphism then*

$$\sum_{e,f \in \mathbb{Z}/p^i\mathbb{Z}} \lambda'\left(\frac{\epsilon f^2 - e^2}{1 + p^j f}\right) = (-p)^i.$$

Proof. Change variables:

$$f' = \frac{f}{\sqrt{1+p^j f}}, \qquad e' = \frac{e}{\sqrt{1+p^j f}},$$

where the square root is taken having remainder $+1 \pmod p$. The desired sum is equal to

$$\sum_{e', f' \in \mathbb{Z}/p^i\mathbb{Z}} \lambda'\left(\epsilon f'^2 - e'^2\right).$$

By counting the number of solutions to the equation $x^2 - \epsilon y^2 = p^k z$, where $k < i$, $z \in (\mathbb{Z}/p^{i-k}\mathbb{Z})^\times$, this sum can be evaluated using Lemmas 5.3 and 5.4, as in the previous subsection. □

Let $\ell = 2m$. We found that every character χ of G with degree $(p-1)p^{\ell-1}$ is induced from a linear character ψ of the subgroup

$$T = \begin{pmatrix} a & \epsilon b + p^m c \\ b & a + p^m d \end{pmatrix} \subseteq G.$$

The following is a list of left coset representatives of T:

$$E_{cd} = \begin{pmatrix} 1 & c \\ & d \end{pmatrix}, \quad 0 \leqslant c < p^m,\ 0 < d < p^m,\ p \nmid d.$$

For $X \in T$, we have

$$\chi(X) = \sum_{c=0}^{p^m-1} \sum_{0<d<p^m, p\nmid d} \dot{\psi}\left(E_{cd} X E_{cd}^{-1}\right)$$

where as usual, $\dot{\psi}$ is the extension of the function ψ which is 0 off T. Assume that $X \notin K_m$, because we have calculated character values on K_m in the previous section. The only conjugacy class type that intersects T is $C_{i\alpha\beta}$. Thus, let $X = \left(\begin{smallmatrix} a & \epsilon p^i b \\ p^i b & a \end{smallmatrix}\right)$, $b \in R^\times$, $0 \leqslant i < m$. We have

$$E_{cd} X E_{cd}^{-1} = \begin{pmatrix} a + p^i bc & p^i b d^{-1}(\epsilon - c^2) \\ p^i bd & a - p^i bc \end{pmatrix}$$

which is in T if and only if $p^m \mid p^i bc$ and $p^m \mid p^i b(d+1)(d-1)$. This is the condition for $\dot{\psi} \neq 0$.

First consider the case $i = 0$. Then $\dot{\psi} = 0$ unless $c = 0$, $d = 1$ or $c = 0$, $d = p^m - 1$, so that

$$\begin{aligned} \chi(X) &= \psi(X) + \psi\left(E_{0(p^m-1)} X E_{0(p^m-1)}^{-1}\right) \\ &= \psi(X) + \psi\left(\begin{pmatrix} 1 & \\ & -1 \end{pmatrix} X \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}\right) \\ &= \psi\begin{pmatrix} a & \epsilon b \\ b & a \end{pmatrix} + \psi\begin{pmatrix} a & -\epsilon b \\ -b & a \end{pmatrix}. \end{aligned}$$

The second-last equality uses the fact that ψ is a class function on T.

Henceforth assume that $i > 0$. The values of d such that $\dot{\psi} \neq 0$ are $p^{m-i} f \pm 1$ for $0 \leqslant f < p^i$. The $+$ and $-$ alternatives are interchanged when $\left(\begin{smallmatrix} a & \epsilon p^i b \\ p^i b & a \end{smallmatrix}\right)$ is replaced with $\left(\begin{smallmatrix} a & -\epsilon p^i b \\ -p^i b & a \end{smallmatrix}\right)$; therefore

$$\chi(X) = \sum_{e,f=0}^{p^i-1} \psi\left(E_{(p^{m-i}e)(1+p^{m-i}f)} \begin{pmatrix} a & \epsilon p^i b \\ p^i b & a \end{pmatrix} E^{-1}_{(p^{m-i}e)(1+p^{m-i})} \right)$$
$$+ \sum_{e,f=0}^{p^i-1} \psi\left(E_{(p^{m-i}e)(1+p^{m-i}f)} \begin{pmatrix} a & -\epsilon p^i b \\ -p^i b & a \end{pmatrix} E^{-1}_{(p^{m-i}e)(1+p^{m-i})} \right).$$

It suffices to compute the first sum because the second is similar. The first is equal to

$$\sum_{e,f=0}^{p^i} \psi\left[\begin{pmatrix} 1 & \\ & 1+p^{m-i}f \end{pmatrix} \begin{pmatrix} 1 & p^{m-i}e \\ & 1 \end{pmatrix}, \begin{pmatrix} a & \epsilon p^i b \\ p^i b & a \end{pmatrix} \right] \psi \begin{pmatrix} a & \epsilon p^i b \\ p^i b & a \end{pmatrix},$$

using the convention

$$[x, y] = xyx^{-1}y^{-1}.$$

We find that (modulo p^{2m}),

$$[E_{cd}, X] = \left[\begin{pmatrix} 1 & \\ & 1+p^{m-i}f \end{pmatrix} \begin{pmatrix} 1 & p^{m-i}e \\ & 1 \end{pmatrix}, \begin{pmatrix} a & \epsilon p^i b \\ p^i b & a \end{pmatrix} \right]$$
$$= \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} + \frac{p^m}{a^2 - \epsilon p^{2i} b^2} \begin{pmatrix} p^i \epsilon b^2 f & -p^i \epsilon b^2 e - \frac{b(\epsilon a f + p^{m-i} a e^2)}{1+p^{m-i}f} \\ abf + p^i b^2 e & -p^i \epsilon b^2 f - abe \end{pmatrix}.$$

This commutator is in K_m, so we can describe its image under ψ in terms of an appropriate matrix $A = \left(\begin{smallmatrix} \alpha & \epsilon\beta \\ \beta & \alpha \end{smallmatrix}\right) \in \mathrm{GL}(2, \mathbb{Z}/p^m\mathbb{Z})$ and linear character λ of $\mathbb{Z}/p^m\mathbb{Z}$ as

$$\psi\big([E_{cd}, X]\big) = \lambda\big(\mathrm{tr}\big(p^{-m} A\big([E_{cd}, X] - 1\big)\big)\big)$$
$$= \lambda\left(\frac{\beta}{a^2 - \epsilon p^{2i} b^2} \left(-p^i \epsilon b^2 e - \frac{b(\epsilon a f + p^{m-i} a e^2)}{1 + p^{m-i} f} \right) + \frac{\epsilon\beta}{a^2 - \epsilon p^{2i} b^2} \big(abf + p^i b^2 e\big) \right)$$
$$= \lambda\left(\frac{\beta ab}{a^2 + \epsilon p^{2i} b^2} p^{m-i} \frac{\epsilon f^2 - e^2}{1 + p^{m-i} f} \right)$$
$$= \lambda'\left(\frac{\epsilon f^2 - e^2}{1 + p^{m-i} f} \right)$$

if we define

$$\lambda' = \lambda^{p^{m-i} ab/(a^2 - \epsilon p^{2i} b^2)},$$

an injective linear character of $\mathbb{Z}/p^i\mathbb{Z}$. Lemma 5.5 now applies and we have

$$\chi(C_{iab}) = (-p)^i \left(\psi \begin{pmatrix} a & \epsilon p^i b \\ p^i b & a \end{pmatrix} + \psi \begin{pmatrix} a & -\epsilon p^i b \\ -p^i b & a \end{pmatrix} \right). \tag{5.3}$$

5.3. Remaining values when $\ell = 2m+1$ is odd

Let Θ denote the induced representation affording θ as in Section 3.3.2. The following result will help us to find the character values of ψ on T in Section 5.4.2.

Lemma 5.6. *For $s \in S \cap K_1$, $s = \begin{pmatrix} 1+px & py\epsilon \\ py & 1+px \end{pmatrix}$ then $\Theta(s) = \lambda(2py\epsilon)I$ where I is the identity matrix.*

Proof. Coset representatives of H in N_m are given by

$$\left\{ n(k) = \begin{pmatrix} 1 & p^m k \\ 0 & 1 \end{pmatrix} \colon 0 \leqslant k < p \right\}.$$

Suppose that $s = \begin{pmatrix} 1+px & py\epsilon \\ py & 1+px \end{pmatrix} = \begin{pmatrix} 1+px & 0 \\ 0 & 1+px \end{pmatrix} + \begin{pmatrix} 0 & py\epsilon \\ py & 0 \end{pmatrix}$. Then

$$\begin{aligned} n(k)^{-1} s n(k) &= \begin{pmatrix} 1+px & 0 \\ 0 & 1+px \end{pmatrix} + \begin{pmatrix} 1 & -p^m k \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & py\epsilon \\ py & 0 \end{pmatrix} \begin{pmatrix} 1 & p^m k \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1+px & 0 \\ 0 & 1+px \end{pmatrix} + \begin{pmatrix} -p^{m+1} yk & py\epsilon \\ py & p^{m+1} yk \end{pmatrix}. \end{aligned}$$

This belongs to $N_{m+1} \subset H$, and $\lambda(n(k)^{-1} s n(k)) = \lambda(2py\epsilon)$. Then $\Theta(s) = \lambda(2py\epsilon) I_p$. $\square$

From Section 3.3.2, $\chi = \mathrm{Ind}_T^G(\psi)$ where ψ is extended from the character $\theta = \mathrm{Ind}_H^{N_m} \phi''$, and ϕ'' is an extension of the character ϕ' of N_{m+1}. Other extensions of ϕ' to a character of H have the form $\phi''\alpha$ where α is a character of H/N_m. Each $\phi''\alpha$ is a component of the restriction of θ to H. So $\theta|_H = \sum_\alpha \phi''\alpha$ where the sum is over the p irreducible characters α of H/N_m. It follows that

$$\psi|_{N_{m+1}} = p\phi'_A \quad \text{and} \quad \psi|_{N_m - N_{m+1}} = 0.$$

Now we will calculate the character values of $\chi = \mathrm{Ind}_T^G \psi \in \mathrm{Irr}(G)$. Similar to the even case, we have the same left coset representatives E_{cd} and we only need to calculate $\chi(X)$ where

$$X = \begin{pmatrix} a & \epsilon p^i b \\ p^i b & a \end{pmatrix}, \quad b \in R^\times,\ 0 \leqslant i < m+1.$$

We have

$$E_{cd} X E_{cd}^{-1} = \begin{pmatrix} a + p^i bc & p^i b d^{-1}(\epsilon - c^2) \\ p^i bd & a - p^i bc \end{pmatrix}.$$

This time, the condition for $\dot{\phi} \neq 0$ yields that $p^{m+1} \mid p^i bc$ and $p^{m+1} \mid p^i b(d+1)(d-1)$.

For the case $i = 0$, we have the same argument as in the even case and we find that

$$\chi(X) = \psi \begin{pmatrix} a & \epsilon b \\ b & a \end{pmatrix} + \psi \begin{pmatrix} a & -\epsilon b \\ -b & a \end{pmatrix}.$$

Assume that $i > 0$. We have

$$\chi(X) = \sum_{e,f=0}^{p^i-1} \psi\left(E_{(p^{m+1-i}e)(1+p^{m+1-i}f)} \begin{pmatrix} a & \epsilon p^i b \\ p^i b & a \end{pmatrix} E^{-1}_{(p^{m+1-i}e)(1+p^{m+1-i}f)}\right)$$
$$+ \sum_{e,f=0}^{p^i-1} \psi\left(E_{(p^{m+1-i}e)(1+p^{m+1-i}f)} \begin{pmatrix} a & -\epsilon p^i b \\ -p^i b & a \end{pmatrix} E^{-1}_{(p^{m+1-i}e)(1+p^{m+1-i}f)}\right).$$

To evaluate the first sum, notice that

$$E_{(p^{m+1-i}e)(1+p^{m+1-i}f)} \begin{pmatrix} a & \epsilon p^i b \\ p^i b & a \end{pmatrix} E^{-1}_{(p^{m+1-i}e)(1+p^{m+1-i})} \in N_{m+1} \quad \text{and} \quad \psi|_{N_{m+1}} = p\psi',$$

we factor out p and use the same method as in the even case, because ϕ' is a homomorphism. Finally, we will get

$$\chi(C_{iab}) = (-p)^i\left(\psi\begin{pmatrix} a & \epsilon p^i b \\ p^i b & a \end{pmatrix} + \psi\begin{pmatrix} a & -\epsilon p^i b \\ -p^i b & a \end{pmatrix}\right), \tag{5.4}$$

which is the same formula as the even case.

5.4. Character values of ψ on T

In last section, we have the formula for $\chi(C_{iab})$ and notice that it depends on ψ where ψ is the corresponding character on the stabilizer T. In this section, we will consider the character values of ψ.

5.4.1. Even case, i.e. $\ell = 2m$

Use the notation of Section 3.2.2. We extend ϕ_A to ψ of $T = K_m S$ in the following two steps.

First, we extend ϕ_A to ϕ'_A of $K_m(K_1 \cap S)$. In order to do this, we only need to extend $\phi_A|_{K_m \cap S}$ to ϕ' of $K_1 \cap S$. Since $\frac{|K_1 \cap S|}{|K_m \cap S|} = p^{2m-2}$, there are p^{2m-2} extensions. Notice that

$$K_1 \cap S = (K_m \cap S)\left\langle\begin{pmatrix} 1+pa & \\ & 1+pa \end{pmatrix}\right\rangle\left\langle\begin{pmatrix} 1 & p\epsilon \\ p & 1 \end{pmatrix}\right\rangle,$$

we only need to define ϕ' on $\langle\begin{pmatrix} 1+pa & \\ & 1+pa \end{pmatrix}\rangle$ and $\langle\begin{pmatrix} 1 & p\epsilon \\ p & 1 \end{pmatrix}\rangle$.

Since

$$C = \begin{pmatrix} 1+pa & \\ & 1+pa \end{pmatrix}^{p^{m-1}} \in K_m \cap S,$$

we can define ϕ' such that

$$\phi'\begin{pmatrix} 1+pa & \\ & 1+pa \end{pmatrix} = \sqrt[p^{m-1}]{\phi_A(C)}.$$

Similarly,

$$D = \begin{pmatrix} 1 & p\epsilon \\ p & 1 \end{pmatrix}^{p^{m-1}} \in K_m \cap S \quad \Rightarrow \quad \phi' \begin{pmatrix} 1 & p\epsilon \\ p & 1 \end{pmatrix} = \sqrt[p^{m-1}]{\phi_A(D)}.$$

This way, we essentially extend ϕ_A to $K_m(K_1 \cap S)$ and there are indeed p^{2m-2} extensions.

Secondly, we want to extend ϕ' to ψ of $K_m S$. From Lemma 5.1,

$$K_m S = K_m(K_1 \cap S)\langle s_0 \rangle \quad \text{and} \quad K_m(K_1 \cap S) \cap \langle s_0 \rangle = 1.$$

Define $\psi|_{K_m(K_1\cap S)} = \phi'$ and $\psi(s_0^i) = \zeta^i$ where ζ is a (p^2-1) root of unity; then ψ is an extension of ϕ_A.

5.4.2. Odd case, i.e. $\ell = 2m+1$

Let $t = \left(\begin{smallmatrix} a & \epsilon p^i b \\ p^i b & a \end{smallmatrix}\right)$ in the class C_{iab}. In the even case, it is clear that $\psi(t)$, occurring in formula (5.3), is a root of unity, since ψ has degree 1. In the odd case, even though ψ has degree > 1, we will show that $\psi(t)$ is again a root of unity, so that (5.4) is essentially independent of whether ℓ is even or odd.

If $i > 0$, then $t \in N_m$. Since

$$\psi|_{N_{m+1}} = p\phi', \qquad \psi(n) = 0 \quad \text{if } n \in N_m,\ n \notin N_{m+1}$$

we know the character value $\phi(t)$ if $t \in N_m$. So suppose that $t \notin N_m$, that is, $i = 0$.

In Section 3.3.2 we constructed ψ this way:

$$\begin{array}{ccccccccc} K_{m+1} & \longrightarrow & N_{m+1} & \longrightarrow & H & \longrightarrow & N_m & \longrightarrow & T \\ \phi_A & \xrightarrow{\text{ext}} & \phi' & \xrightarrow{\text{ext}} & \phi'' & \xrightarrow{\text{ind}} & \theta & \xrightarrow{\text{ext}} & \psi \end{array}$$

where

$$N_i = K_i(K_1 \cap S)\left\{\begin{pmatrix} a & \\ & a \end{pmatrix}\right\}, \qquad H = N_{m+1}\left\langle \begin{pmatrix} 1+p^m & \\ & 1 \end{pmatrix} \right\rangle.$$

Note that

$$N_m = \left\{ \begin{pmatrix} t + p^m x + pa & p^m y + pb\epsilon \\ pb & t + pa \end{pmatrix} \right\}.$$

Coset representatives of N_{m+1} in N_m are given by

$$\left\{ g(x,y) = \begin{pmatrix} 1 + p^m x & p^m y \\ 0 & 1 \end{pmatrix} : 0 \leqslant x, y < p \right\}$$

and the coset $g(x,y)N_{m+1}$ is equal to

$$\left\{ \begin{pmatrix} t + p^m x + p^{m+1} d + pa & p^m y + p^{m+1} e + pb\epsilon \\ pb & t + pa \end{pmatrix} \right\}.$$

It follows that an element $\left(\begin{smallmatrix} r & s \\ u & v \end{smallmatrix}\right)$ of N_m is in the coset $g(x,y)N_{m+1}$ precisely when

$$(r - v)/p^m \equiv x \mod p \quad \text{and} \quad (s - \epsilon u)/p^m \equiv y \mod p. \tag{5.5}$$

Lemma 5.7. *Suppose that $t \in T$ and $t \notin N_m$. Then $|\psi(t)| = 1$.*

Proof. Let Θ be the representation of N_m having character $\theta = \operatorname{Ind}_H^{N_m} \phi''$, and let Ψ be the representation of T having character ψ, so $\operatorname{Res}_{N_m}^T \Psi = \Theta$.

Since Θ is irreducible, then the $\mathbb{C}$-span of the set $\{\Theta(n)\colon n \in N_m\}$ is equal to the full set $M_p(\mathbb{C})$ of $p \times p$ matrices over $\mathbb{C}$. If $n \in N_m$, then n belongs to some coset $g(x, y)N_{m+1}$, and $n = g(x, y)n'$ for some $n' \in N_{m+1}$. By Lemma 5.6, $\Theta(n') = \alpha I$ for some scalar α. It follows that $\Phi(n)$ is a scalar multiple of $\Phi(g(x, y))$, and that the set $\{\Theta(g(x, y))\colon 0 \leqslant x, y < p\}$ is a generating set of $M_p(\mathbb{C})$. A dimension count shows that $\{\Theta(g(x, y))\colon 0 \leqslant x, y < p\}$ is a $\mathbb{C}$-basis of $M_p(\mathbb{C})$.

The group T acts on $M_p(\mathbb{C})$ with $t \in T$ acting on a matrix B by $\Psi(t)^{-1} B \Psi(t)$, giving a representation of T of degree p^2 having character $\overline{\psi}\psi$.

Given $t = \left(\begin{smallmatrix} a & b\epsilon \\ b & a \end{smallmatrix}\right) \in T$, $t \notin N_m$, suppose that $t^{-1} g(x, y)t$ belongs to the coset $g(x, y)N_m$. We claim that $(x, y) = (0, 0)$. We have

$$\begin{aligned} t^{-1} g(x, y)t &= \frac{1}{a^2 - b^2\epsilon} \begin{pmatrix} a & -b\epsilon \\ -b & a \end{pmatrix} \left(I + p^m \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix}\right) \begin{pmatrix} a & b\epsilon \\ b & a \end{pmatrix} \\ &= I + \frac{p^m}{a^2 - b^2\epsilon} \begin{pmatrix} a^2x + aby & ab\epsilon x + a^2 y \\ -abx - b^2 y & -b^2\epsilon x - aby \end{pmatrix}. \end{aligned}$$

The criterion (5.5) that this belongs to $g(x, y)N_{m+1}$ gives

$$\begin{aligned} a^2x + aby + b^2\epsilon x + aby &\equiv \left(a^2 - b^2\epsilon\right)x \mod p, \\ ab\epsilon x + a^2 y + ab\epsilon x + b^2\epsilon y &\equiv \left(a^2 - b^2\epsilon\right)y \mod p. \end{aligned}$$

Since $t \notin N_m$, then b is not divisible by p. So we can divide by $2b$, and get

$$\begin{aligned} b\epsilon x + ay &\equiv 0 \mod p, \\ ax + by &\equiv 0 \mod p. \end{aligned}$$

Since $a^2 - b^2\epsilon$ is a unit of R, then $x = y = 0$, as claimed.

Thus the action of t on $M_p(\mathbb{C})$ permutes the basis $\{\Theta(g(x, y))\colon 0 \leqslant x, y < p\}$ modulo scalars, and the only basis vector fixed (up to scalar multiples) is $g(0, 0)$. It follows that the trace of $\Gamma(t) = 1$. Since the trace of Γ is equal to $\overline{\psi}\psi$, we deduce that $|\psi(t)| = 1$. $\square$

Since $t = \left(\begin{smallmatrix} a & \epsilon b \\ b & a \end{smallmatrix}\right)$, then $t \in S$. We know that $S = (K_1 \cap S)\langle s_0\rangle$, and that for $s \in K_1 \cap S$ we have $\Theta(s) = \alpha I$ for some $\alpha \in \mathbb{C}^\times$. So it suffices to calculate $\psi(s)$ for $s \in \langle s_0\rangle = S_0$.

Lemma 5.8. *For $s \in S_0$, the value of $\psi(s)$ is a root of unity.*

Proof. We have $\psi|_T = \theta = \operatorname{Ind}_H^T \phi''$. We have shown that there exists ϕ'' which is trivial on $Z = \left\{\left(\begin{smallmatrix} a & \\ & a \end{smallmatrix}\right)\right\}$. We will first prove the lemma assuming that $\theta = \operatorname{Ind}_H^T \phi''$ for this particular ϕ''.

Suppose Φ is the representation of T corresponding to ϕ. Note that $T = N_m S_0$; this is not a semidirect product, since s_0^{p+1} has the form $\left(\begin{smallmatrix} a & \\ & a \end{smallmatrix}\right)$. Define the function σ on T by

$$\sigma(t) = \begin{cases} 1, & \text{if } t \in N_m, \\ \det(\Phi(t)) & \text{if } t \in S_0. \end{cases}$$

Since $\Phi(s_0^{p+1}) = I$, we see that σ is a character of T with N_m in its kernel. Let $\Phi_1 = \Phi\sigma$; then $\det(\Phi_1(s_0)) = (\det(\Phi(s_0)))^{p+1} = 1$. Since $\Phi(s_0^{p+1})$ is the identity, then $\det(\Phi_1(s)) = 1$ for all $s \in S_0$.

We claim that ϕ_1 is the unique extension of θ to T with the property that a representation Φ_1 affording ϕ_1 satisfies $\det(\Phi_1(s_0)) = 1$. If we have a different extension ϕ_2 to T of θ with this property, then $\phi_2 = \phi_1\alpha$ where α is a non-trivial irreducible character of T/N_m. Let Φ_2 be the corresponding representation; we have

$$\Phi_2(s_0) = \Phi_1(s_0)\alpha(s_0) \quad \Rightarrow \quad \det\bigl(\Phi_2(s_0)\bigr) = \alpha(s_0)^p \neq 1.$$

So ϕ_1 is indeed unique with the property $\det(\Phi_1(s_0)) = 1$. For $s \in S_0$, $\phi(s)$ is a sum of powers of μ where μ is a primitive $p+1$ root of unity. Let $\tau \in \mathrm{Aut}(\mathbb{Q}(\mu) : \mathbb{Q})$. Since N_m/Z is a p-group, then $\theta^\tau = \theta$, so ϕ_1^τ is another extension of θ with $\det(\Theta^\tau(s_0)) = 1$. By uniqueness of ϕ_1, $\phi_1^\tau = \phi_1$. Hence $\phi_1(s) \in \mathbb{Q}$. From Lemma 5.7, $|\phi_1(s)| = 1$, so $\phi_1(s_0) = \pm 1$.

The general case now follows, since different characters ψ are obtained by multiplication by a root of unity. □

6. Characters of degree $(p^2-1)p^{\ell-2}$ and the character values

In this section, we will find the character values of the irreducible characters of degree $(p^2-1)p^{\ell-2}$. Similarly to the last section, we first find character values on $K_{\ell-i}$, $1 \leqslant i \leqslant \frac{\ell}{2}$, and then work on the remaining character values in two cases depending on whether ℓ is even or odd.

6.1. Character values of elements in $K_{\ell-i}$, $1 \leqslant i \leqslant \frac{\ell}{2}$

Let A be the matrix $\left(\begin{smallmatrix} & p^j\beta \\ 1 & \end{smallmatrix}\right)$ over the ring $R_i = \mathbb{Z}/p^i\mathbb{Z}$ where $\beta \in R_i^\times$, $1 \leqslant j \leqslant i$. Let ϕ_A be the corresponding character of $K_{\ell-i}$:

$$\phi_A \begin{pmatrix} 1+p^{\ell-i}a & p^{\ell-i}b \\ p^{\ell-i} & 1+p^{\ell-i}a \end{pmatrix} = \lambda\left(\operatorname{tr}\begin{pmatrix} 0 & p^j\beta \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ 1 & a \end{pmatrix}\right) = \lambda\bigl(b + p^j\beta\bigr).$$

Lemma 6.1. *The following list of cardinality $p^{2i-2}(p^2-1)$ includes exactly one representative from each right coset of $\left\{\left(\begin{smallmatrix} w & p^j\beta y \\ y & w \end{smallmatrix}\right)\right\} \subset \mathrm{GL}(2, R_i)$:*

$$\left\{\begin{pmatrix} 1 & \\ & d \end{pmatrix}\begin{pmatrix} 1 & c \\ & 1 \end{pmatrix} : c \in R_i,\ d \in R_i^\times\right\}, \quad \left\{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & \\ & d \end{pmatrix}\begin{pmatrix} 1 & pc \\ & 1 \end{pmatrix} : pc \in R_i,\ d \in R_i^\times\right\}.$$

Proof. It is easy to check that any two matrices from the list are non-conjugate to each other. Since the index of $\left\{\left(\begin{smallmatrix} w & p^j\beta y \\ y & w \end{smallmatrix}\right)\right\}$ in $\mathrm{GL}(2, R_i)$ is $p^{2i-2}(p^2-1)$, we have all the coset representatives. □

Suppose χ is any irreducible character of G whose restriction to K_{l-i} contains copies of ϕ_A. Similarly to Section 5.1, let $X = 1 + p^{\ell-i}\left(\begin{smallmatrix} a & b \\ 1 & a \end{smallmatrix}\right)$, by Clifford's theorem, we have

$$\chi(X) = \frac{\deg\chi}{p^{2i-2}(p^2-1)} \sum_{c \in R_i, d \in R_i^\times} \lambda\left(\operatorname{tr}\left(A\frac{E_{cd}XE_{cd}^{-1}}{p^{\ell-i}}\right)\right) + \sum_{pc \in R_i, d \in R_i^\times} \lambda\left(\operatorname{tr}\left(A\frac{F_{cd}XF_{cd}^{-1}}{p^{\ell-i}}\right)\right),$$

where

$$E_{cd} = \begin{pmatrix} 1 & \\ & d \end{pmatrix}\begin{pmatrix} 1 & c \\ & 1 \end{pmatrix}, \qquad F_{cd} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & \\ & d \end{pmatrix}\begin{pmatrix} 1 & pc \\ & 1 \end{pmatrix}.$$

By calculation, we have

$$\operatorname{tr}\left(A\left(\frac{E_{cd}XE_{cd}^{-1}}{p^{\ell-i}}\right)\right)=p^j\beta d+d^{-1}\left(b-c^2\right),\qquad \operatorname{tr}\left(A\frac{F_{cd}XF_{cd}^{-1}}{p^{\ell-i}}\right)=p^j\beta d^{-1}\left(b-p^2c^2\right)+d.$$

Therefore

$$\chi(X)=p^{\ell-2i}\left[\sum_{c\in R_i,d\in R_i^\times}\lambda\left(p^j\beta d+d^{-1}\left(b-c^2\right)\right)+\sum_{pc\in R_i,d\in R_i^\times}\lambda\left(p^j\beta d^{-1}\left(b-p^2c^2\right)+d\right)\right].$$

Denote $P=\sum_{c\in R_i,d\in R_i^\times}\lambda(p^j\beta d+d^{-1}(b-c^2))$, we have the following results:

(i) for all b, $\sum_{pc\in R_i,d\in R_i^\times}\lambda(p^j\beta d^{-1}(b-p^2c^2)+d)=\left\{\begin{pmatrix}0, & \text{if } i>j,\\ -p^{i-1}, & \text{if } i=j\end{pmatrix}\right.$.

(ii) If b is a non-square unit, then $P=\left\{\begin{pmatrix}0, & \text{if } i>1,\\ -p, & \text{if } i=1\end{pmatrix}\right.$. If $b=u^2$, $u\in R_i^\times$, then $P=\left\{\begin{pmatrix}0, & \text{if } i>1,\\ p, & \text{if } i=1\end{pmatrix}\right.$.

It is trivial when $i=1$. So we assume $i>1$. In this case, $x=p^j\beta d+d^{-1}(b-c^2)$ can attain any unit in R_i and the number that each unit appears is the same. Since $\sum_{y\in R_i^\times}\lambda(y)=0$, we only care about the values of x such that $p\mid x$. Notice that

$$p\mid x\quad\Rightarrow\quad p\mid\left(b-c^2\right)\quad\Rightarrow\quad b=u^2,\quad u\in R_i^\times,$$

we have if b is non-square, then x is always in $R_i^\times$. Therefore, $P=0$.

Now if b is a square unit, then x can attain 0. Say $p^j\beta d_1^2+b-c_1^2=0$. Let $d_2=d_1$, $c_2^2=c_1^2-npd_2$, then $p^j\beta d_2+d_2^{-1}(b-c_2^2)=pn$. Namely, x can attain pn, $\forall n$, and the number that each pn appears is the same. Since $\sum_{pn\in R_i}\lambda(pn)=0$, the result follows.

(iii) $b=p^kn$, $k<j$, $p\nmid n$.

In this case, x can attain p^km where $p\nmid m$ and the number that x attains each p^km is $\frac{(p^i-p^{i-1})p^{i-1}}{p^{i-k}-p^{i-k-1}}=p^{k+i-1}$. Hence,

$$P=p^{k+i-1}\sum_{m\in(\mathbb{Z}/p^{i-k}\mathbb{Z})^\times}\lambda\left(p^km\right)=\left\{\begin{pmatrix}-p^{k+i-1}, & \text{if } k=i-1,\\ 0, & \text{if } k<i-1\end{pmatrix}\right..$$

(iv) $b=p^jn$, where $-\beta n$ is a square unit.

x takes $p^{j+1}m, m\in\mathbb{Z}/p^{i-j-1}\mathbb{Z}$ in this case. And the number that x takes each above value is $\frac{p^{2i-2}(p-1)}{p^{i-j-1}}=p^{i+j-1}(p-1)$. Therefore,

$$P=p^{i+j-1}(p-1)\sum_{m\in\mathbb{Z}/p^{i-j-1}\mathbb{Z}}\lambda\left(p^{j+1}m\right)=\left\{\begin{pmatrix}p^{i+j-1}(p-1), & \text{if } i=j+1,\\ 0, & \text{if } i>j+1\end{pmatrix}\right..$$

(v) Otherwise, x takes p^jm, $m\in(\mathbb{Z}/p^{i-j}\mathbb{Z})^\times$ and the number is $\frac{p^{2i-2}(p-1)}{p^{i-j}-p^{i-j-1}}=p^{i+j-1}$. This implies

$$P = p^{i+j-1} \sum_{m \in (\mathbb{Z}/p^{i-j}\mathbb{Z})^\times} \lambda(p^j m) = \left\{ \begin{pmatrix} -p^{i+j-1}, & \text{if } i = j+1, \\ p^{i+j}, & \text{if } i = j, \\ 0, & \text{if } i > j+1 \end{pmatrix} \right. .$$

From the above results, we can determine the character values of χ on each kind of conjugacy class.

6.2. Remaining values for characters with degree $(p^2-1)p^{\ell-2}$

In this section, we want to evaluate the remaining character values. We will first work on the even case, then the odd case will follow similarly.

We can extend ϕ_A to ψ of T, and $\chi = \mathrm{Ind}_T^G \psi \in \mathrm{Irr}(G)$. We can first define ψ satisfying

$$\psi \begin{pmatrix} 1+p^m a & p^m b \\ p^{m-j} c & 1+p^m d \end{pmatrix} = \lambda(p^m b + p^m c\beta)$$

and get different extensions by multiplying roots of unity.

Pick coset representatives of T in G to be

$$E_{cd} = \begin{pmatrix} 1 & \\ & d \end{pmatrix} \begin{pmatrix} 1 & c \\ & 1 \end{pmatrix}, \quad 0 \leqslant c,\ d < p^m,\ d \in R^\times,$$

and

$$F_{cd} = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \begin{pmatrix} 1 & \\ & d \end{pmatrix} \begin{pmatrix} 1 & pc \\ & 1 \end{pmatrix}, \quad 0 \leqslant pc,\ d < p^m,\ d \in R^\times.$$

Notice that the only conjugacy class type that intersects T is $B_{i\alpha\beta}$, so we only need to evaluate the character values of χ on $X = \left(\begin{smallmatrix} a & p^{i+j}\beta \\ p^i & a \end{smallmatrix}\right)$, $0 \leqslant i < m$. By calculation, we have

$$F_{cd} X F_{cd}^{-1} \notin T, \quad \text{for all } c, d,$$

and

$$E_{cd} X E_{cd}^{-1} = \begin{pmatrix} a + p^i c & p^i d^{-1}(p^j \beta - c^2) \\ p^i d & a - p^i c \end{pmatrix}.$$

First, we assume $m > i + j$. Then in order for $E_{cd} X E_{cd}^{-1} \in T$, we must have

$$c = p^{m-i} e, \quad d = p^{m-i-j} f \pm 1, \quad 0 \leqslant e < p^i,\ 0 \leqslant f < p^{i+j}.$$

Since

$$\begin{aligned} Y &= E_{(p^{m-i}e)(1+p^{m-i-j}f)} X E^{-1}_{(p^{m-i}e)(1+p^{m-i-j}f)} \begin{pmatrix} a & -p^{i+j}\beta \\ -p^i & a \end{pmatrix} \\ &= (a^2 - p^{2i+1}\beta) \begin{pmatrix} 1 + p^m * & d^{-1}(-p^{2m-i}a^{-1}e^2 - p^{m+i+j}\beta e a^{-2} - p^m \beta^{-1} f) \\ p^{m-j} f a^{-1} + p^{m+i} e a^{-2} & 1 + p^m * \end{pmatrix}, \end{aligned}$$

we have

$$\psi(Y)=\psi\begin{pmatrix} a^2-p^{2i+1}\beta & \\ & a^2-p^{2i+1}\beta \end{pmatrix}\lambda\left[p^{2m-i-j}a^{-1}\left(1+p^{m-i-j}f\right)^{-1}\left(\beta f^2-p^je^2\right)\right].$$

Notice that

$$\psi(X)\psi\begin{pmatrix} a & -p^{i+j}\beta \\ -p^i & a \end{pmatrix}=\psi\begin{pmatrix} a^2-p^{2i+1}\beta & \\ & a^2-p^{2i+1}\beta \end{pmatrix},$$

we know

$$\begin{aligned}&\psi\left(E_{(p^{m-i}e)(1+p^{m-i-j}f)}XE^{-1}_{(p^{m-i}e)(1+p^{m-i-j}f)}\right)\\ &\quad=\psi\begin{pmatrix} a & p^{i+1}\beta \\ p^i & a \end{pmatrix}\lambda\left[p^{2m-i-j}a^{-1}\left(1+p^{m-i-j}f\right)^{-1}\left(\beta f^2-p^je^2\right)\right].\end{aligned}$$

Similarly,

$$\begin{aligned}&\psi\left(E_{(p^{m-i}e)(p^{m-i-j}f-1)}XE^{-1}_{(p^{m-i}e)(p^{m-i-j}f-1)}\right)\\ &\quad=\psi\begin{pmatrix} a & -p^{i+j}\beta \\ -p^i & a \end{pmatrix}\lambda\left[p^{2m-i-j}a^{-1}\left(p^{m-i-j}f-1\right)^{-1}\left(\beta f^2-p^je^2\right)\right].\end{aligned}$$

Note that $1+p^{m-i-j}f$ and $1-p^{m-i-j}f$ are two square units, we can make a substitution,

$$f'=\frac{f}{\sqrt{1+p^jf}},\qquad e'=\frac{e}{\sqrt{1+p^jf}},$$

to get

$$\begin{aligned}\chi\begin{pmatrix} a & p^{i+j}\beta \\ p^i & a \end{pmatrix}&=\sum_{e=0}^{p^i-1}\sum_{f=0}^{p^{i+j}-1}\left\{\psi\begin{pmatrix} a & p^{i+j}\beta \\ p^i & a \end{pmatrix}\lambda\left[p^{2m-i-j}a^{-1}\left(\beta f^2-p^je^2\right)\right]\right\}\\ &\quad+\sum_{e=0}^{p^i-1}\sum_{f=0}^{p^{i+j}-1}\left\{\psi\begin{pmatrix} a & -p^{i+j}\beta \\ -p^i & a \end{pmatrix}\lambda\left[p^{2m-i-j}a^{-1}\left(p^je^2-\beta f^2\right)\right]\right\}.\end{aligned}$$

The above two summations can be calculated because

$$\begin{aligned}\sum_{e=0}^{p^i-1}\sum_{f=0}^{p^{i+j}-1}\lambda\left[p^{2m-i-j}a^{-1}\left(\beta f^2-p^je^2\right)\right]&=\sum_{e=0}^{p^i-1}\lambda\left(-p^{2m-i}a^{-1}e^2\right)\sum_{f=0}^{p^{i+j}-1}\lambda\left(p^{2m-i-j}a^{-1}\beta f^2\right)\\ &=\sum_{e=0}^{p^i-1}\lambda_1\left(e^2\right)\sum_{f=0}^{p^{i+j}-1}\lambda_2\left(f^2\right),\end{aligned}$$

for some injective homomorphisms

$$\lambda_1:\mathbb{Z}/p^i\mathbb{Z}\to C^\times,\qquad \lambda_2:\mathbb{Z}/p^{i+j}\mathbb{Z}\to C^\times.$$

Since we have the results that

$$\sum_{e=0}^{p^{2k}-1} \lambda\big(e^2\big) = p^k, \qquad \sum_{e=0}^{p^{2k+1}-1} \lambda\big(e^2\big) = p^k G(\lambda),$$

where

$$G(\lambda) = \sum_{r=0}^{p-1} \lambda\big(r^2\big)$$

is a quadratic Gauss sum which has a formula depending on λ and p, we can evaluate the summations in the formula of $\chi\left(\begin{smallmatrix} a & p^{i+j}\beta \\ p^i & a \end{smallmatrix}\right)$.

Now if $i+j \geqslant m$, we have $E_{cd} X E_{cd}^{-1} \in T \Rightarrow c = p^{m-i}e$, $d \in R^\times$. Therefore,

$$\psi\big(E_{cd} X E_{cd}^{-1}\big) = \psi\begin{pmatrix} a & \\ & a \end{pmatrix} \lambda\big[a^{-1}\big(p^{i+j}\beta d + p^{i+j}\beta d^{-1} - p^{2m-i}e^2 d^{-1}\big)\big],$$

and we have

$$\chi(X) = \psi\begin{pmatrix} a & \\ & a \end{pmatrix} \sum_{0 \leqslant e < p^i, d \in (\mathbb{Z}/p^m\mathbb{Z})^\times} \lambda\big[a^{-1}\big(p^{i+j}\beta d + p^{i+j}\beta d^{-1} - p^{2m-i}e^2 d^{-1}\big)\big].$$

Denote $P = \sum_{0 \leqslant e < p^i, d \in (\mathbb{Z}/p^m\mathbb{Z})^\times} \lambda[a^{-1}(p^{i+j}\beta d + p^{i+j}\beta d^{-1} - p^{2m-i}e^2 d^{-1})]$, in order to evaluate P we have the following 3 cases.

(i) $i+j > 2m-i$.

In this case, we can find $\lambda' : \mathbb{Z}/p^i\mathbb{Z} \to \mathbb{C}^\times$ such that

$$\begin{aligned} P &= \sum_{e \in \mathbb{Z}/p^i\mathbb{Z}, d \in (\mathbb{Z}/p^m\mathbb{Z})^\times} \lambda'\big[p^{2i+j-m}\beta\big(d + d^{-1}\big) - e^2 d^{-1}\big] \\ &= p^{m-i} \sum_{e \in \mathbb{Z}/p^i\mathbb{Z}, d \in (\mathbb{Z}/p^i\mathbb{Z})^\times} \lambda'\big[p^{2i+j-m}\beta d + d^{-1}\big(p^{2i+j-m}\beta - e^2\big)\big]. \end{aligned}$$

Now P can be evaluated because the above summation has been done in Section 6.1.

(ii) $i+j = 2m-i$.

In this case

$$P = p^{m-i} \sum_{e \in \mathbb{Z}/p^i\mathbb{Z}, d \in (\mathbb{Z}/p^i\mathbb{Z})^\times} \lambda'\big[\beta d + d^{-1}\big(\beta - e^2\big)\big].$$

This summation can be evaluated using the same arguments as in Section 5.1.

(iii) $i+j < 2m-i$.

Now we can find injective $\lambda_1 : \mathbb{Z}/p^{2m-i-j}\mathbb{Z} \to \mathbb{C}^\times$, $\lambda_2 : \mathbb{Z}/p^i\mathbb{Z} \to \mathbb{C}^\times$ to simplify P such that

$$\begin{aligned} P &= \sum_{d\in(\mathbb{Z}/p^m\mathbb{Z})^\times} \lambda_1\big(d+d^{-1}\big) \sum_{e\in\mathbb{Z}/p^i\mathbb{Z}} \lambda_2\big(e^2 d^{-1}\big) \\ &= p^{i+j-m} \sum_{d\in(\mathbb{Z}/p^{2m-i-j}\mathbb{Z})^\times} \lambda_1\big(d+d^{-1}\big) \sum_{e\in\mathbb{Z}/p^i\mathbb{Z}} \lambda_2\big(e^2 d^{-1}\big). \end{aligned}$$

Note that the second summation above is a quadratic Gauss sum and we have

$$\sum_{e\in\mathbb{Z}/p^{2k}\mathbb{Z}} \lambda\big(e^2\big) = p^k$$

for any injective $\lambda : \mathbb{Z}/p^{2k} \to \mathbb{C}^\times$, thus when i is even, we have

$$P = p^{\frac{i}{2}} p^{i+j-m} \sum_{d\in(\mathbb{Z}/p^{2m-i-j}\mathbb{Z})^\times} \lambda_1\big(d+d^{-1}\big).$$

In the case that $2m-i-j=1$ this is a Kloosterman sum. It seems likely that we cannot simplify this sum any further.

6.2.1. Odd case

Define

$$\phi_A : K_{m+1} \to \mathbb{C}^\times, \qquad \phi_A\big(I + p^{m+1}B\big) = \lambda\big(\mathrm{tr}\big(p^{m+1}AB\big)\big) = \lambda\big(p^{m+1}b + p^{m+j+1}c\beta\big).$$

Then

$$T = \mathrm{Stab}_G(\phi_A) = \left\{\begin{pmatrix} a & p^j\beta b + p^m c \\ b & a + p^m d\end{pmatrix}\right\} = K_m S, \quad \text{where } S = \left\{\begin{pmatrix} w & p^j\beta y \\ y & w\end{pmatrix}\right\}.$$

Recall that we have the following picture

$$\begin{array}{ccccccccc} K_{m+1} & \longrightarrow & N & \longrightarrow & T_0 & \longrightarrow & T & \longrightarrow & G \\ \phi_A & \xrightarrow{\text{ext}} & \phi'_A & \xrightarrow{\text{ext}} & \phi' & \xrightarrow{\text{ind}} & \psi & \xrightarrow{\text{ind}} & \chi \end{array},$$

where $N = \{\left(\begin{smallmatrix} 1+p^m a & p^{m+1}b \\ p^{m+1-j}c & 1+p^m d\end{smallmatrix}\right)\}$, and we can extend ϕ_A to ϕ'_A of N such that

$$\phi'_A\begin{pmatrix} 1+p^m a & p^{m+1}b \\ p^{m+1-j}c & 1+p^m d\end{pmatrix} = \lambda\big(p^{m+1}b + p^{m+1}c\beta\big).$$

Similarly to the even case, we can get different extensions ϕ'_A by multiplying roots of unity.

Let $T_0 = \{\left(\begin{smallmatrix} a & p^j\beta b + p^{m+1}c \\ b & a+p^m d\end{smallmatrix}\right)\} = NS$, then $N \lhd T_0$ and ϕ'_A is stable under T_0. Thus, we can extend ϕ'_A of N to ϕ' of T_0 such that ϕ' is trivial on the center of G. Since $T_0 \lhd T$, and $\mathrm{Stab}_T(\phi') = T_0$, we have $\psi = \mathrm{Ind}_{T_0}^T \phi' \in \mathrm{Irr}(T)$. Therefore, $\chi = \mathrm{Ind}_T^G \psi = \mathrm{Ind}_{T_0}^G \phi' \in \mathrm{Irr}(G)$. In order to evaluate the character values of χ, we can consider χ as induced from ϕ' of T_0. The coset representatives of T_0 are

$$E_{cd} = \begin{pmatrix} 1 & \\ & d \end{pmatrix} \begin{pmatrix} 1 & c \\ & 1 \end{pmatrix}, \quad 0 \leqslant d < p^m,\ d \in R^\times,\ 0 \leqslant c < p^{m+1},$$

and

$$F_{cd} = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \begin{pmatrix} 1 & \\ & d \end{pmatrix} \begin{pmatrix} 1 & pc \\ & 1 \end{pmatrix}, \quad 0 \leqslant c, d < p^m,\ d \in R^\times.$$

Let $X = \left(\begin{smallmatrix} a & p^{i+j}\beta \\ p^i & a \end{smallmatrix}\right)$, $0 \leqslant i \leqslant m$, then

$$F_{cd} X Fcd^{-1} \notin T', \quad \text{for all } c, d, \quad \text{and} \quad E_{cd} X E_{cd}^{-1} = \begin{pmatrix} a + p^i c & p^i d^{-1}(p^j\beta - c^2) \\ p^i d & a - p^i c \end{pmatrix}.$$

We first assume that $m + 1 > i + j$; then in order for $E_{cd} X E_{cd}^{-1} \in T_0$, we must have

$$c = p^{m-i} e, \quad d = p^{m+1-i-j} f \pm 1, \quad 0 \leqslant e < p^{i+1},\ 0 \leqslant f < p^{i+j-1}.$$

By the same method as in the even case, we first calculate

$$Y = E_{(p^{m-i}e)(1+p^{m+1-i-j}f)} X E^{-1}_{(p^{m-i}e)(1+p^{m+1-i-j}f)} \begin{pmatrix} a & -p^{i+j}\beta \\ -p^i & a \end{pmatrix}$$

and deduce that

$$\phi'(Y) = \phi' \begin{pmatrix} a^2 - p^{2i+1}\beta & \\ & a^2 - p^{2i+1}\beta \end{pmatrix} \lambda\bigl[p^{\ell-i-j} a^{-1} \bigl(1 + p^{m+1-i-j} f\bigr)^{-1} \bigl(p^j \beta f^2 - e^2\bigr)\bigr].$$

Denote $\phi'\left(\begin{smallmatrix} a & p^{i+j}\beta \\ p^i & a \end{smallmatrix}\right) = P^+$, $\phi'\left(\begin{smallmatrix} a & -p^{i+j}\beta \\ -p^i & a \end{smallmatrix}\right) = P^-$, we have

$$\begin{aligned} &\phi'\bigl(E_{(p^{m-i}e)(1+p^{m+1-i-j}f)} X E^{-1}_{(p^{m-i}e)(1+p^{m+1-i-j}f)}\bigr) \\ &\quad = P^+ \lambda\bigl[p^{\ell-i-j} a^{-1} \bigl(1 + p^{m+1-i-j} f\bigr)^{-1} \bigl(p^j \beta f^2 - e^2\bigr)\bigr]. \end{aligned}$$

Similarly,

$$\begin{aligned} &\phi'\bigl(E_{(p^{m-i}e)(p^{m+1-i-j}f-1)} X E^{-1}_{(p^{m-i}e)(p^{m+1-i-j}f-1)}\bigr) \\ &\quad = P^- \lambda\bigl[p^{\ell-i-j} a^{-1} \bigl(p^{m+1-i-j} f - 1\bigr)^{-1} \bigl(p^j \beta f^2 - e^2\bigr)\bigr]. \end{aligned}$$

Making a substitution gives us

$$\begin{aligned} \chi \begin{pmatrix} a & p^{i+j}\beta \\ p^i & a \end{pmatrix} = {} & \sum_{e=0}^{p^{i+1}-1} \sum_{f=0}^{p^{i+j-1}} \bigl\{P^+ \lambda\bigl[p^{\ell-i-j} a^{-1} \bigl(p^j \beta f^2 - e^2\bigr)\bigr]\bigr\} \\ & + \sum_{e=0}^{p^{i+1}-1} \sum_{f=0}^{p^{i+j-1}} \bigl\{P^- \lambda\bigl[p^{\ell-i-j} a^{-1} \bigl(e^2 - p^j \beta f^2\bigr)\bigr]\bigr\}. \end{aligned}$$

Once again, the above summations can be written as a product of two Gauss sums. Like the even case, if $i + j \geqslant m + 1$, we have

$$\chi(X) = \phi\begin{pmatrix} a & \\ & a \end{pmatrix} \sum_{0 \leqslant e < p^{i+1}, d \in (\mathbb{Z}/p^m\mathbb{Z})^\times} \lambda\left[a^{-1}\left(p^{i+j}\beta d + p^{i+j}\beta d^{-1} - p^{2m-i}e^2 d^{-1}\right)\right].$$

This sum can be discussed in the same way as the even case.

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